Alternative Starting Point for Teaching Fractions

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We investigate the viability of a new approach to initial fraction instruction. We establish the need to empirically investigate whether the proposed approach shares the strengths of currently used approaches, specifically, whether students will (a) construe problems based on the proposed approach as experientially real, and (b) bring up ideas that could be build upon in subsequent fraction instruction. We then present an analysis of sixteen student interviews from a school in southern Mexico (ages 8 and 9). The analysis supports the conjecture that the proposed approach to initial fraction instruction can be viable, and thus warrants further research attention.

Introduction

In this paper, we build on Freudenthal's (1983) analysis of the fraction concept to propose what we came to view as a viable alternative starting point for fraction instruction. The alternative consists of regarding *comparing*, instead of *fracturing*, as the primary activity from which students are expected to make sense of fractions. It involves using, from the start, problems aimed at orienting students to reason about unit fractions as quantities that account for the relative size of magnitude values,[1] rather than for the size of pieces generated by equally partitioning a food item such as a cake, a candy-bar, a pizza, or a loaf of bread. An example of the problems that, we argue, represent the alternative starting point, orients students to reason about how much milk a cup holds, if so many cups of the same capacity would hold as much milk as a milk carton.

Theoretical Framing

Starting Point for Instruction

Our perspective on instructional design builds on the RME theory (e.g., Gravemeijer, 1994). Central to our perspective is the construct of *hypothetical learning trajectory* (HLT). An important step in formulating a HLT involves choosing a *viable starting point* for instruction. Such a starting point consists of problems that have the potential of fulfilling three characteristics: (a) become experientially real to students, (b) trigger informal ways of reasoning that can be a basis for developing increasingly sophisticated mathematical ways of knowing, and (c) serve as paradigmatic cases in which to "anchor students' increasingly abstract mathematical activity" (Cobb, et al., 1997, p. 159). We use these characteristics as a lens to examine two approaches for introducing students to fraction learning.

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[1]In this paper we follow Freudenthal in using the terms *magnitude*, *magnitude value* and *quantity*. M*agnitude* is a measurable property of an object; for instance, *length*. In the literature, what we call *magnitude* has also been referred to with the terms *quantitative property* and *dimension*. *Magnitude value* refers to the specific size of a magnitude in an object; for instance, the length of a stick. It has been referred to as *quantitative magnitude* (Thompson & Saldanha, 2003) and *quantity not quantified* (Lamon, 2007). Finally, we use the term *quantity* to refer to the measure of the value of a magnitude that is expressed with a number; for instance, the length of a stick is 12 (centimeters). It has been referred to as *measure*.

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Revisiting Freudenthal's Insights on Fraction

In his didactical phenomenology, Freudenthal (1983) distinguished between two different ways of conceiving fractions: *as fracturers* and *as comparers*. In the first of these two ways fractions are conceived as numbers that account for actions that physically modify objects. A notorious example of such actions is partitioning a pizza into equal sized pieces. In contrast, in the second approach, fractions are numbers that account for the relative size of magnitude values. An example is the capacity of a cup to hold milk, relative to the capacity of a milk carton. The actions that take place in order to conduct such comparisons—pouring milk and marking milk levels—serve to gauge a specific property of the objects involved, and need not affect the cohesiveness of the objects.

Fraction as Fracturer

For Freudenthal, the fraction as fracturer approach involves construing the meaning of a unitary fraction as a piece of a whole, which is obtained by equal partition. In this approach, the unit-whole is typically understood as being an object; that is, as a thing (e.g., an actual candy bar) instead of as a property of thing (e.g., the mass or the length that is specific to a certain candy bar). A fraction is understood as something that is produced by fracturing the object, literally. In this interpretation, learners would understand 1/5 of a candy bar as one of five equal pieces of what used to be a single candy bar (see Figure 1).

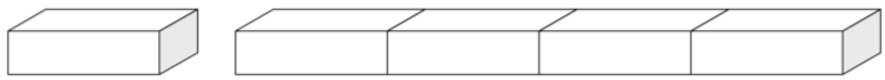

Figure 1. A fifth as a fraction of a unit-whole (i.e., a candy bar), where setting it apart implies disassembling the unit-whole.

The typical instructional situations that are used to introduce fractions to students involve the equal partition of a food item—such as a pizza—so that it can be shared among a certain number of people. From the point of view of a designer with already developed fraction reasoning, the food item is meant to serve as a representation of a unit-whole, of a continuous magnitude (i.e., a mass of food embodying the value of one), and the size of the pieces produced by equally partitioning it, as the entities that unit-fractions quantify. However, the fraction 1/b is also meant to be interpreted by learners, in a realistic way, as the *part* of a food item (e.g., a pizza) that someone will get when receiving 1 out of b equal-sized pieces (Clarke & Roche, 2009). In the case of the fraction a/b (a>1), the expectation is that students will come to make sense of the numerator as a number-symbol that accounts for a "number of parts of that name or size" (Clarke & Roche, 2009, p. 136).

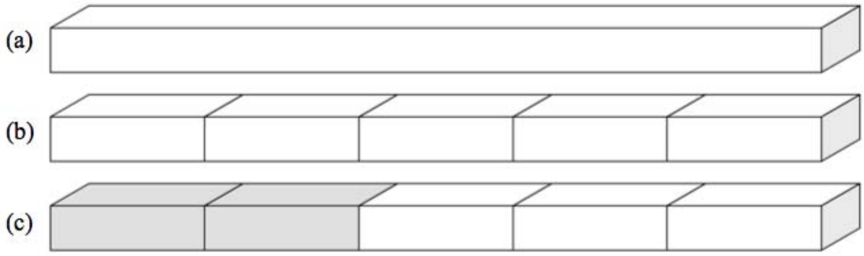

Figure 2. Fractions as numbers that account for equal partitions. From top to bottom: (a) a candy bar as unit-whole, (b) the candy bar partitioned into fifths (i.e., five equal pieces), and (c) 2/5 of the candy bar shaded.

Situations involving the partitioning and sharing of food-items have been widely used as a context for supporting students in making sense of fractions. These situations appear in textbooks, in specialised teacher-literature, and in research (e.g., Steffe & Olive, 2010; Streefland, 1991; Tzur, 1999). The wide use of these situations can be (at least partially) justified by revisiting the three characteristics of a viable starting point. Much evidence exists that students can readily and meaningfully engage with the food partitioning situations, even from an early age (cf. Pitkethly & Hunting, 1996). In addition, these situations have been shown to trigger informal ways of reasoning, on which students can be supported to make sense of some important fraction relations, such as the relative size of unitary fractions (e.g., 1/3>1/4), and basic fraction equivalencies (e.g., 1/2=2/4; 1/4=2/8; cf. Lamon, 2007).

However, it is with regard to the third criterion—serving as an anchor for students' reasoning as learning evolves—that Freudenthal's analysis brings into question the viability of situations involving the equal partition and sharing of food items as starting points for instruction. Freudenthal (1983) judged situations, in which learners operate on objects, that is *fraction as fracturer* situations, to be "much too restricted not only phenomenologically but also mathematically" (p. 144). For him, they are phenomenologically restricted because fractions become consistently "seen or imagined within the whole" (p. 147). As a consequence, in the fracturer approach, fractions become restricted to being "proper fractions only (<1)" (p. 147). In addition, these situations are mathematically restricted because the operation of partition as the basis of equivalence relation always only generates a restricted number of elements in each equivalence class.[2] As Freudenthal explains, in order to use fractions as a phenomenological source of the rational number, "an equivalence of broader scope is needed, as well as the unrestricted availability of objects in every equivalence class" (p. 147).

Freudenthal's assessment of the shortcomings of fraction as fracturer is consistent with the concerns expressed by Thompson and Saldanha (2003) about the limitations of traditional fraction instruction:

> Students are often instructed, and therefore learn, that the fractional part is contained within the whole, so "A is some fraction of B" connotes a sense of inclusion to them, that A is a subset of B. As a result, statements like "A is 6/5 of B" make no sense to them (p. 107).

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[2]In instructional terms, a pizza can always only produce four fourths, which makes imagery of 7/4 problematic. This problem is inherent to how a magnitude is constituted in a system of quantities via operation of partition: if equivalence is defined only by the operation of partition, we have no means to compare quantities of elements produced from different *unit-wholes* (different pizzas).

These authors also contend that students' oftentimes develop an image of fractions, in which they are seen as numbers that account for *so many out of so many*. Thompson and Saldanha (2003) contend that this image too

> …possesses a sense of inclusion—that the first "so many" must be included in the other "so many". As a result, they [the students] will not accept the idea that we can speak of one quantity's size as being a fraction of another's size when they have nothing physically in common. They will accept "The number of boys is what fraction of the number of children?", but they will be puzzled by "The number of boys is what fraction of the number of girls?" (p. 104).

The concrete imagery that learners develop while working on problems that involve food partitioning and sharing is not consistent with, and thus does not easily lend itself to building more sophisticated fraction and rational number imagery. These situations do not fulfil the third criterion of a viable starting point, serving as an anchor for students' reasoning as learning evolves. Instead, they orient students to develop an initial understanding of fractions as fracturer. As studies on learning difficulties in fractions document, this initial understanding makes it particularly difficult for children to later make sense of fractions as numbers that soundly quantify values bigger than one, as well as the size of something in relation to something else that does not contain it (Hackenberg, 2007). We thus consider it reasonable to regard these situations as inadequate in initial fraction instruction and to seek a viable alternative.

An Alternative to Equal Partitioning

Freudenthal (1983) and Thompson and Saldanha (2003) envisioned a similar alternative to the use of equal partitioning—an alternative in which *ratio comparisons* (i.e., fraction as comparer) become the focus of fraction instruction. In this alternative, fractions are used to quantify magnitude values by comparing them to a magnitude value of reference (construed as having the size of 1), in multiplicative terms (Figure 3).

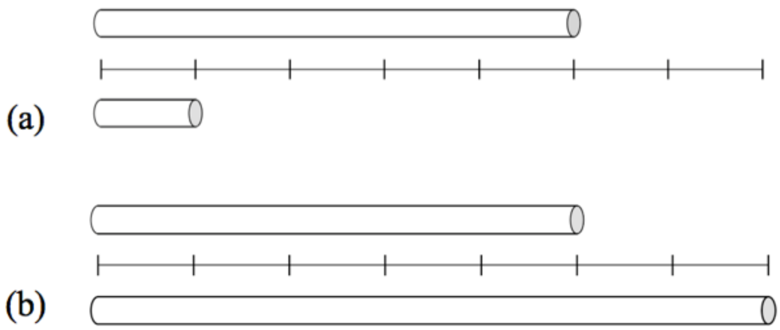

Figure 3. (a) The unit whole (a stick) construed as a specific length (i.e., a magnitude value), and 1/5 of it as another specific length that satisfies the condition of 5 iterations of it being as long as the unit-whole. (b) 7/5 construed as a specific length that is 7 times as long as 1/5 of the length of the unit-whole.

Based on the analyses of these authors, it is possible to picture an alternative image of a unitary fraction to cultivate in students. It becomes a magnitude value that satisfies an iterative condition, regarding a magnitude value of reference, in the following terms: A is $1/n$ as large as B when B is n times as large as A.

Regarding the three characteristics of a viable starting point for instruction, the analyses suggest that opting for problems aimed at orienting students to interpret unitary fractions as multiplicands would fulfill the third criterion. However, it is unclear whether such problems would fulfill the first two criteria of a viable starting point. We report on initial insights into viability of the approach with regard to the first two criteria.

Methodology and Data Sources

We conducted clinical interviews with sixteen students (ages 8 and 9) who formed the only third-grade classroom in a public school in southern Mexico. An analysis of their notebooks suggested that they had had limited opportunities to learn about fractions with understanding. Instead, their prior instructional experiences seemed to have entailed much of what Anyon (1981) identified as work that most often involves "substantial amounts of rote activity" (p. 203). In addition, students in this classroom performed poorly in the national standardised test known as ENLACE. This group was thus suitable for testing the viability of the approach as a starting point.

Interview Protocol

For the iterative approach to be instructionally useful, we considered that students should, from the outset, have access to ways of reasoning about a core notions that pupils are typically expected to make sense of in the initial phases of fraction instruction, relative size of unitary fractions (NCTM, 2000).

The interview protocol included three problems. In the first, *Milk Carton*, the students were asked to make judgments about the relative capacity of seven different kinds of cups, based on how many cups of each kind could be filled with the milk contained in a carton. The actual cups were not shown to them (Figure 4). We asked students to make comparisons similar to: 1/2 vs. 1/4; 1/3 vs. 1/4; 1/7 vs. 1/9; 1/20 vs. 1/1 (formulated "Which cup can hold more milk, plastic or glass one?").

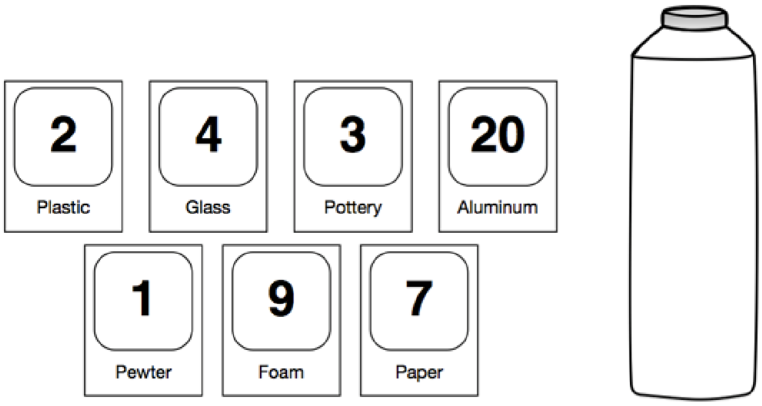

Figure 4. Cards used in the *Milk Carton* problem to show how many cups of each kind could be filled with the milk contained in a carton and image of the milk carton that was used.

Two more similar problems, *Kangaroos,* and *Water Tanks* were introduced and will be reported on during the conference presentation. In these, the students were encouraged to make context-based fraction comparisons such as 1/2 vs. 1/4 vs. 1/5, (*Kangaroos*) and ?/8 = 1/2, ?/8 = 1/4, ?/8 = 3/4 (*Water Tanks*).

Data Coding

The interview data was open coded, following an ongoing formulation of provisional categories (Strauss & Corbin, 1990). We sought to identify similarities and differences in how students construed the problems, and in the mathematical reasoning that emerged.

The evidence we used for determining that a problem had been construed as *experientially real* (first characteristic of a viable starting point) consisted of verbal expressions and gestures that suggested that a student was reasoning about the quantities involved in the problem, and not just about the numbers. For instance, in the *Milk Carton* problem, this type of evidence included students referring to *amounts of milk* or making gestures with their hands indicating the *size of cups*. The *informal ways of reasoning* about fractions (second characteristic) that the students developed in response to the interview problems are outlined in the Results section.

Results

Given the evidence criteria, all of the students construed the three problems as experientially real. The following extract illustrates Marilu reasoning about the size of the plastic cup (Figure 4):

> Interviewer: When I serve one cup of milk, how far does the milk carton empty?
> Marilu: To here [marking the carton at about the middle].
> Interviewer: Why?
> Marilu: Because they are this size [gesturing with her fingers the size of a cup]

Gestures referring to the actual size of cups indicated to us that a student like Marilu had interpreted the problem as involving actual quantities. In other words, it indicated that a student was imagistically involved with the problem at hand and, thus, engaging in personally meaningful mathematical activity. Results from this component of the analysis support considering the proposed starting point for fraction instruction as viable. They suggest that it would be possible to support a group of third grade students, like those we interviewed, to readily construe problems based on the *comparer* approach as experientially real.

We further documented the informal ways of reasoning students developed about *relative size of unitary fractions* and about *basic fraction equivalencies*. Here we report on instructionally relevant differences in how students reasoned about *relative size of unitary fractions*, while solving problems where magnitude values were defined as multiplicands that satisfy an iterative condition (Table 1).

Table 1
Number of Students in Different Categories, According to How They Reasoned about the Relative Size of Unitary Fractions

	Category 1 The bigger the number, the larger the magnitude value	Category 2 Visual evidence	Category 3 Coming to anticipate	Category 4 Anticipating that the more iterations, the smaller the magnitude values
N	2	3	5	6

Students that showed the least sophisticated reasoning (Category 1, N=2) had some intuitions that were consistent with assessing the size of magnitude values relative to how many iterations of it would be necessary to complete a unit. These students correctly regarded the aluminum cups (twentieths) as being small, and the pewter cups (ones) as being big. However, these intuitions were not sufficiently robust to allow pupils to make sound comparisons of the sizes of magnitude values. They instead followed a different rationale: The bigger the number, the larger the magnitude value.

Category 2 students (N=3), relying on the marks of their estimates (e.g., on the milk carton), made comparisons consistent with the idea that the more iterations of a magnitude value were needed to make as much as a unit, the smaller the magnitude value had to be. However, during the interviews, these students did not come to understand this principle sufficiently well to consider that it would apply in every case.

Category 3 students (N=5) came to anticipate that the more iterations of a magnitude value that were needed to make as much as a unit, the smaller the magnitude value had to be. Finally, the remaining students (Category 4, N=6) readily judged that the more iterations of a magnitude were needed to make as much as a unit, the smaller the magnitude value had to be.

To illustrate, Zaide was one of the students in category 3. She initially anticipated that the plastic cups (halves) would be smaller than the glass cups (fourths). After marking on the carton sensible estimates of where the levels of the milk would be if one plastic cup and one glass cup were served, respectively, she changed her mind and considered that the plastic cups would be bigger. When comparing the glass cups (fourths) and the pottery cups (thirds), she followed a similar path. Finally, she anticipated that the foam cups (ninths) would hold less milk than the paper cups (sevenths):

Interviewer: Which would be bigger?
Zaide: The foam cups [ninths].
Intervewer: Bigger?
Zaide: Oh, no, the paper ones [sevenths].
Interviewer: The paper or the foam ones?
Zaide: Paper.
Interviwer: Why the paper ones?
Zaide: Because if they put nine cups it goes down less. And if you put seven it goes down faster.
Interviewer: Which can hold more?
Zaide: The seven ones [paper cups].
Interviewer: The seven ones can hold more?
Zaide: Because they are not many cups.

In the excerpt it is noticeable that Zaide reasoned about the respective capacity of the paper cups based on how the levels of the milk in the carton would change as these cups were served. Her previous estimations seemed to have helped her develop an image that when more cups were to be served, the level of the milk in the carton would drop less with one serving ("it goes down less"). This image allowed her to correctly anticipate that the foam cups (ninths) would be smaller than the paper cups (sevenths). Importantly, as Zaide progressed through *Kangaroos* and *Water Tanks* problems, she came to readily anticipate the relative comparisons correctly.

With different levels of sophistication, the large majority of the students correctly judged the relative size of the magnitude values involved in the different problems. Even the students who showed the least sophisticated reasoning had some intuitions that were consistent with the just mentioned property of unitary fractions (e.g., regarded the aluminum cups (twentieths) as being small, and the pewter cups (ones) as being big).

Of particular significance were the Category 3 students, whose reasoning became more sophisticated *during* the interview. They started by using a natural number rationale in comparing the capacity of the cups, and later came to correctly conceive their relative capacity, without having to rely on visual evidence. Their case suggests that problems based on the iterative approach can be a means by which to support students to engage in progressive mathematisation (Gravemeijer & Doorman, 1999) in the fraction realm.

Conclusion and Significance

The results indicate that problems based on the iterative approach can be a viable starting point for fraction instruction. Students' informal reasoning during the interviews was consistent with coming to make sense of two core notions of initial fraction instruction—relative size of unitary fractions and basic fraction equivalencies. These findings provide a strong justification for developing instructional sequences that take advantage of the identified starting point.

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