Variation and Mathematics Pedagogy

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This discussion paper put forwards variation as a theme to structure mathematical experience and mathematics pedagogy. Patterns of variation from Marton's Theory of Variation are understood and developed as types of variation interaction that enhance mathematical understanding. An idea of a discernment unit comprising mutually supporting variation interactions is proposed and used as a building block for a mathematics pedagogy that resembles a developmental sequence from coarse idea to precise definition. Classification of plane figures is used as a pedagogical example for illustration.

Mathematical Experiences and Variation

What is mathematics teaching and learning? I take the perspective that teaching and learning of mathematics is about providing learners opportunities to experience mathematics and to create (new) mathematical experiences. How does a learner experience a mathematical "thing"? For examples, memorize a formula, execute an algorithm, write down a string of mathematical symbols, prove a proposition, recognize a pattern, etc. Are all these indications that a learner has experienced mathematics? These activities are no doubt related to mathematics, but are they critical enough to ensure the occurrence of a genuine mathematical experience? To address this question, I may ask the metaphysical question of what mathematics is, but such a discussion would diverse away from pedagogy. Rather, I take a more pragmatic path to ask how to empower a learner to experience mathematics.

Consider an experience as a bridge to connect a person to an observable phenomenon (directly or indirectly), then concerning pedagogy, this would mean how to make an epistemic connection between a learner and an object of learning (the "thing" to be learnt). In the mathematics classroom, this connectedness could be achieved, for example, through dynamic feedback processes that occur between a learner and an object of learning under a designed usage of tools like a ruler, grid paper, or even sophisticated ICT teaching and learning environments. In practice, tools or manipulatives are potential mediators for mathematical experience (cf. Maschietto & Trouche, 2010).

Resnick (1997) advocated that mathematics is a "science of pattern". Pattern can be interpreted as an emerging invariant structure when a phenomenon is undergoing changes or *variation*. Leung (2010) described mathematical experience as "the discernment of invariant pattern concerning numbers and/or shapes and the re-production or re-presentation of that pattern." Variation is about what changes, what stays constant and what the underlying rule is.

> "To become aware of what is constant in the flux of nature and life is the first step in abstract thinking …. The conception of constancy in change provides the first guarantee of meaningful actions." (Wilhelm., 1973, p.23)

In phenomenology, one of the hermeneutic rules is "seek out structural or invariant features of the phenomena", furthermore,

> "The probing activity of investigation is called *variational method*… Variations "possibilize"

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phenomena. Variations thus are devices that seek the invariants in variants and also seek to determine the limits of a phenomenon." (Ihde, 1986, pp. 39-40)

Discernment, variation and simultaneity are the central concepts in the phenomenographic research approach in which learning and awareness are interpreted under a theoretical framework of variation (Marton & Booth, 1997; Marton, Runesson, & Tsui, 2004). The basic ideas behind this approach can be captured in a nutshell by the following:

> "As we always act in relation to situations as we see them, effective actions spring from effective ways of seeing. Seeing a situation in a certain way amounts to discerning those aspects which are critical for engaging in effective action and taking all of them into consideration (focusing on them) at the same time. In order to discern a certain aspect, one must have experienced variation in those aspects. There is no discernment without variation. The only way we can prepare for the un-definable variation in the future is by experiencing variation in the present and by having experienced variation in the past." (The abstract of a seminar given by Ference Marton at The University of Hong Kong, 21 Nov 2006)

According to Marton's Theory of Variation, discernment of critical features occurs under systematic interaction between a learner and the thing to be learnt, and variation is the agent that generates such interaction (Marton, Runesson, & Tsui, 2004). Local variation in different aspects of a phenomenon unveils the invariant structure of the whole phenomenon. Invariants are critical features that define or generalize a phenomenon. This matches nicely with what doing mathematics is about, for a major aim of mathematical activity is to separate out invariant patterns while different mathematical entities are varying, and subsequently to generalize, classify, categorize, symbolize, axiomatize and operationalize these patterns.

Dienes (1963) attributed the abstraction and the generalization processes in mathematical thinking by what he called the perceptual variability principle and the mathematical variability principle:

> "The perceptual variability principle stated that to abstract a mathematical structure effectively, one must meet it in a number of different situations to perceive its purely structural properties. The mathematical variability principle stated that as every mathematical concept involved essential variables, *all* these mathematical variables need to be varied if the full generality of the mathematical concept is to be achieved." (Dienes, 1963, p.158)

Sieving out invariants using variation is thus an essence of experiencing mathematics.

A mathematics pedagogy that is rooted in variation is one that purposefully provides students with means to experience variation through, for example, concrete tools, multiple representation of a concept, strategically designed exercise, etc. in order to create a *mathematically rich learning environment* (Leung, 2010) that allows students to discern invariant. The root of variation is to observe relationships produced between the parts and the whole while focus of attention changes, and to discover what stays unchanged among these relationships. Mathematically speaking, this is liken to discern relationship among contextual variables; that is, possible aspects that can vary in a mathematical situation. Pedagogically, this means the teacher needs to decide when to focus on which variable(s) in order to bring about the intended learning outcome. To probe into how to create variation interactions that foster discernment, I will discuss types of possible variation interaction and using them to develop a model of mathematics pedagogy based on variation that reflects a process of mathematics knowledge acquisition.

Types of Variation Interaction

In Marton' Theory of Variation, four patterns of variation were proposed: contrast, generalization, separation and fusion. They form the kernel for discernment under variation. Leung (2008) used these patterns to develop a lens of variation to interpret explorations in dynamic geometry environments. These patterns of variation are fundamental elements used to organize a variation experience and they generate interactions between learners and the object of learning. In this paper, rather than patterns, I consider them as types of *variation interaction* that could build up mathematical experience.

> A variation interaction is a strategic use of variation to interact with a mathematics learning environment in order to bring about discernment of mathematical structure.

This strategic use can be teacher-designed by or learner-initiated. They are understood as follows.

Contrast is to discern whether something satisfies a certain condition or not, that is, whether something "is" or "isn't". Thus contrast seeks to distinguish different and unlike things. On the one hand, to comprehend a mathematical idea, one often resorts to finding counter-examples in order to discern the critical features of the idea. On the other hand, a mathematical concept can be represented in multiple ways, by contrasting them; one can seek to discern traces of invariant features behind the concept. A major activity in doing mathematics is classification which aims to arrive at invariant concepts and contrast is a classification activity.

Separation is the *awareness of critical features and/or dimensions of variation*. A dimension of variation is an *emerging* feature of a phenomenon which can take on different "values" while some aspects of the phenomenon are varying; it may or may not be an invariant feature depending on the focus of attention and what are varying. For example, when analyzing the nature of sunlight using suitable dispersive prism, *color* can become a dimension of variation which takes on values "red", "green", "blue", etc with the orientation of a prism as a varying aspect. In particular, when an aspect of a phenomenon is varying while the other aspects are being kept fixed, critical features of this aspect of the phenomenon may be discerned and a dimension of variation may be separated out. Mathematically, it is liken to conceiving variables in a physical phenomenon, or to seeing the information encoded in the cross-sections of a solid shape while the cutting plane is varying in a systematic way. Separation is an awareness of part-whole relationship awakened by a systematic refined contrast obtained by purposely varying or not varying certain aspects aiming to differentiate the invariant parts from a whole.

Generalization is a variation interaction that is inductive in nature. When the same invariant pattern appears in different situations under contrast and separation, this pattern may be de-contextualized. Generalization is a purposeful contrast to explore whether an observed pattern can occur while certain aspects vary. It is a verification and conjecture-making activity checking the general validity of a separated out pattern which is often a goal of mathematical exploration.

Fusion integrates critical features or dimensions of variation into a whole under simultaneous co-variation. By fusing the separated-out critical features or dimensions of variation together, a whole concept may appear. It is liken to perceiving the graph of a mathematical function as a representation of a relationship between variable x and variable y. By contrasting critical features and dimensions of variation, fusion sculpts meaning and concept when parts of a whole vary in interconnected ways.

Thus in a nutshell, these types of variation interactions are basically different foci of simultaneous contrast.

Discernment Unit

In a pedagogical situation, these four types of variation interaction act together in a concerted way to bring about discernment. Consider the example of classifying geometrical plane figures as an illustration. A collection of different types of plane figures is given to students. Teacher can design suitable activities asking students to visually sort these figures into groups. The design should then focus on a visual intuitive process of contrast and generalization on the number of sides of the figures, the size and orientation of the figures, and the shapes of the angles, etc. That is, cycles of activity contrasting the different visual features and generalizing by sorting these visual features. These features may become critical features or dimensions of variation, and consequently intuitive types of figures can be separated out and classified. For example, figures with the same number of sides or angles, figures with pairs of equal opposite sides, etc. The activities should be designed in a way such that students are sensitized to becoming aware of critical features of the figures and are given the opportunity of becoming simultaneously aware of the variety of figures (Figure 1).

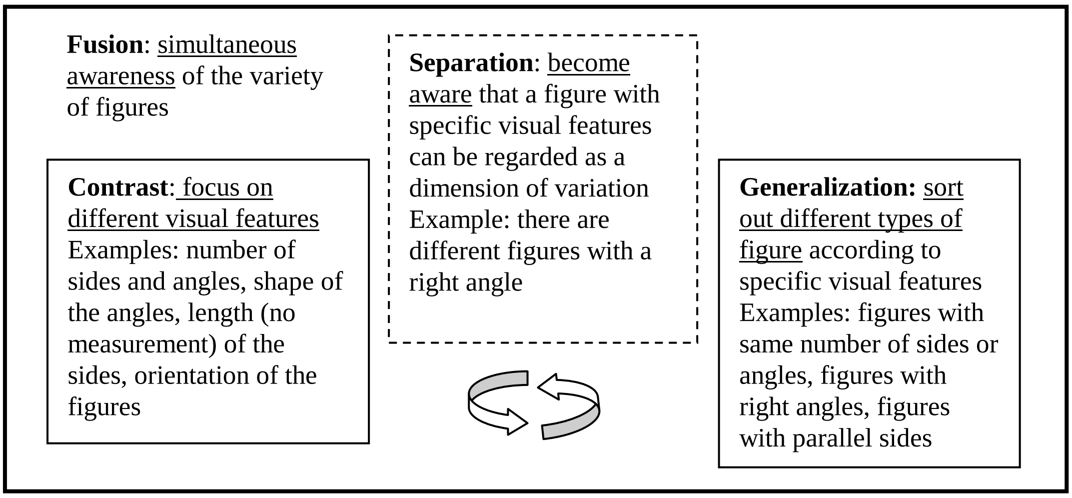


Figure 1. A visual intuitive classification of plane figures utilizing the four types of variation interactions. The circular arrows and the dotted rectangle indicate a mutually enhancing interaction between contrast and generalization is at work to bring about awareness of dimensions of variation and/or critical features

In light of this, I propose an idea of a *discernment unit* which stands for a unit of a pedagogical process driven by these four types of variation interaction. It is a convolution of contrast and generalization driven by separation fused together by simultaneous awareness of critical features (Figure 2). Convolution here means combining the mutually enhancing interaction between contrast and generalization to produce an invariant. It is driven by separation in the sense of becoming aware of different dimensions of variation and/or of critical features via some variation strategies. This discernment unit will be the fundamental building block for a model of mathematics pedagogy based on variation which I will discuss in the next session.

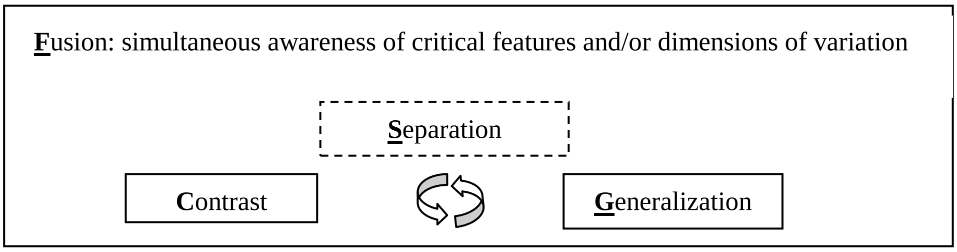


Figure 2. A discernment unit driven by types of variation interaction

Classification of Plane Figures

I have illustrated the use of a discernment unit to frame a mathematical inquiry on visually classifying plane figures. Different focus of attention gives different classification and the types of classification depend on what aspects of a plane figure are being varied. In this sense, a discernment unit is like a function of what's being varied. Let us consider two other possible ways to classify plane figures and see what the discernment unit could generate in these cases. The classification by visual intuition described in the previous section will be labeled as *Discernment Unit One*.

Discernment Unit Two: Classification of Plane Figures by Properties

Students are asked to classify plane figures according to the figures' properties. Students need to separate out these properties through contrast and generalization experiences that involve activities like measuring length of sides and angles, making a list of the figures, and finding (sufficient) conditions to define a figure. These properties can be regarded as dimensions of variation with different figures as values. For example, four right angles is a dimension of variation with squares and rectangles as values, two pairs of opposite equal sides is a dimension of variation with all types of parallelograms as values. The classification is then a fusion of properties that can be used to distinguish different types of figures (Figure 3)

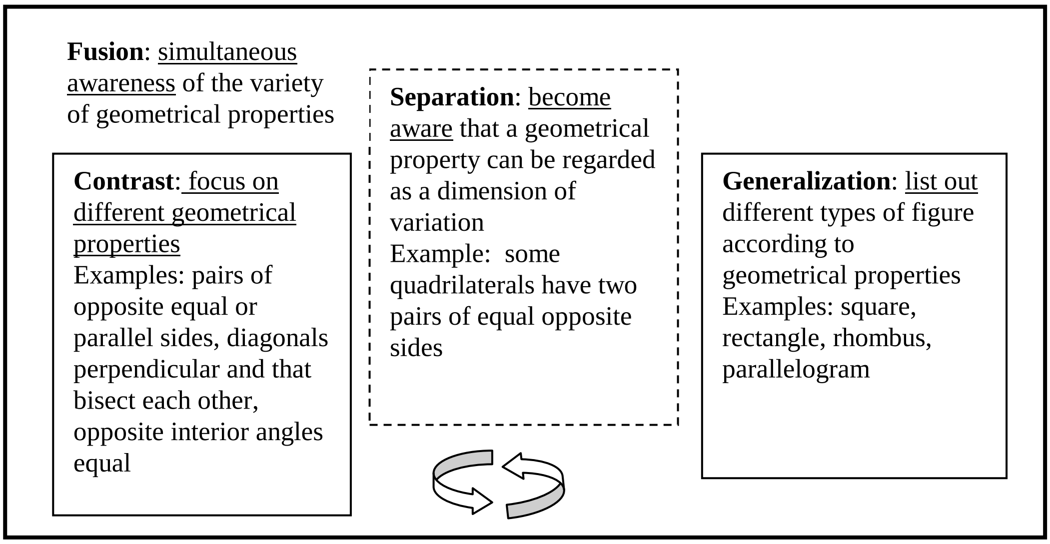


Figure 3. A discernment unit for classification of plane figures by properties

Discernment Box Three: Classification of Plane Figures by Relationships between Properties

Students are expected to use relationships between properties to classify plane figures with activities like looking for counter examples (contrast) to refute the statement "if the diagonals of a quadrilateral are perpendicular to each other, then it must be a square" and investigating the relationship between angles and sides to arrive at the generalization "if a quadrilateral has equal opposite interior angles, then it must be a parallelogram". The dimensions of variation being separated out in Discernment Unit Two (geometrical properties) now become possible values of a dimension of variation which is a refined concept of a figure characterized by equivalent properties. For example, for a parallelogram, "two pairs of equal opposite sides" is equivalent to "two pairs of parallel opposite sides". Thus the concept "parallelogram" is formed as a fusion of properties and their relationships to each other (Figure 4).

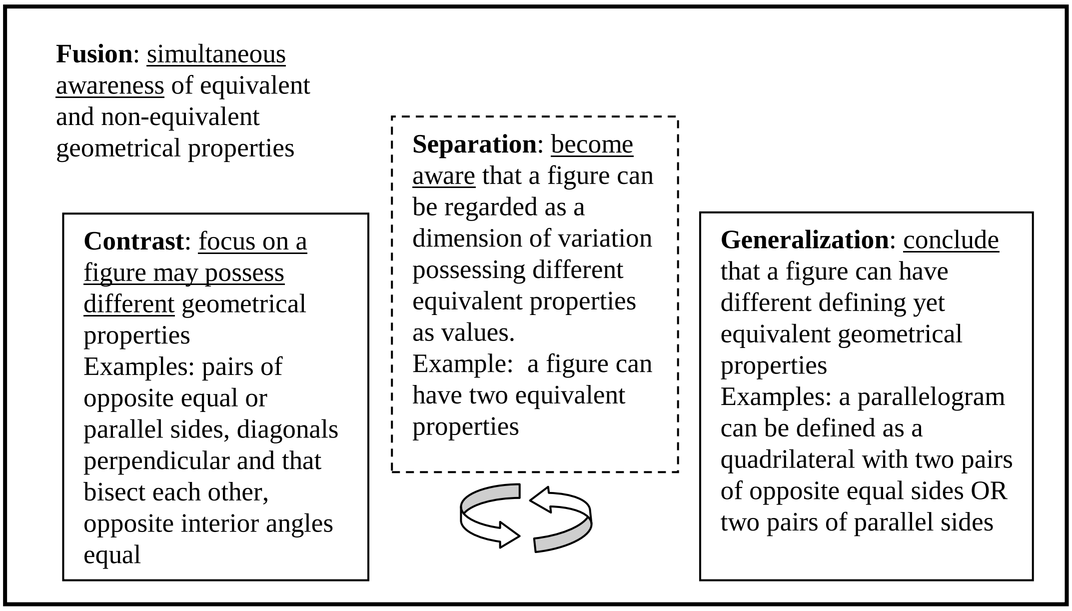


Figure 4. A discernment unit for Classification of plane figures by relationships between properties

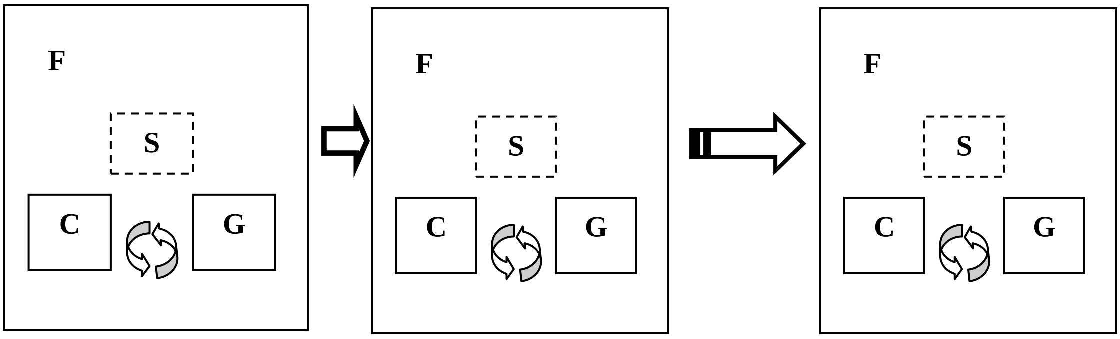
A Model of Mathematics Pedagogy Based on Variation

Discernment Units One, Two and Three form a hierarchy of gradual refined concepts of plane figures starting from the intuitive primitive (visual), to the refined (properties) and finally to the fine-grained (relationship between properties). Teachers who take a constructivist's path like guided-reinvention in Freudenthal's mathematisation framework where mathematics knowledge is acquired through a refinement process of doing suitable realistic mathematical activities (Freudenthal, 1973, 1991) can utilize this hierarchy to design a pedagogical sequence on the understanding of plane figures (Figure 5).

Figure 5. A pedagogical time sequence on the understanding of plane figures

This process of "mathematical understanding" is sequenced by a chain of variation interactions where simultaneity and focus of attention play critical roles. Notice that the process consists of a sequence of variation interactions that is increasing in sophisticated levels of contrast (visual perception, properties, inter-relating properties) reflecting the evolution of an idea starting from a primitive stage to reaching a more formal mathematical stage. The arrows in Figure 5 refer to *shifts of attention* (cf. Mason, 2008) that may cause a refinement of understanding. The first arrow indicates a shift of focus from visual features to geometrical properties and the second arrow from geometrical properties to a figure, and consequently the related dimension of variation changes from "figure" to "property" and back to "figure" again (see Figures 2, 3 and 4). Aside from synchronic simultaneity when students focus on different aspects of plane figure at the same time, diachronic simultaneity plays the critical role of connection of variation experiences gained in previous and present discernment units. In particular, following such a sequence, learning tasks can be designed capitalizing the variation interactions to develop a mathematical concept progressively from coarse description to precise definition. This would empower students with a rich mathematical experience as discussed in the beginning of the paper. Figure 6 presents such a model of mathematics pedagogy based on variation.

Mathematically Primitive



Mathematically Formal

Figure 6. A model of mathematics pedagogy based on variation. **F**(Fusion), **C**(Contrast), **S**(Separation), **G**(Generalization)

Each of the above *discernment units* represents a developing mathematical concept that is fused together by a process of contrast and generalization driven by separation. The sequence represents a process of refinement of the mathematical concept, from primitive to progressively formal and mathematical. This pedagogical model can be seen as a process of doing mathematics and acquiring mathematical knowledge. It is rooted in variation as doing mathematics is basically a variation activity seeking invariant structures in the midst of changes. A learner needs to experience the evolution of a mathematics idea to fully comprehend it, and this pedagogical model can create such an experience.

I hope this discussion opens a window to view variation as a powerful agent to generate mathematical knowledge and stimulates further research and practices using variation as a pedagogical tool in the mathematics classroom.

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