UNDERSTANDING MATHEMATICAL TEXT THROUGH PEER EXPLANATIONS

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This paper reports on research that investigated processes through which students come to understand mathematical text. Transcripts from a Year 11 lesson are analysed to illustrate three contexts in which students interrogated mathematical text: reading assigned by the teacher, spontaneous reading, and reading peer-produced text. Although current moves for mathematics education reform discourage over-reliance on textbooks as a source of knowledge and authority, this study demonstrates that reading as a social practice can stimulate students' critical engagement with mathematical ideas.

Current interest in the role of language and communication in mathematics learning is reflected in curriculum documents such as the National Statement on Mathematics for Australian Schools (Australian Education Council, 1991) and the NCTM Standards (e.g. National Council of Teachers of Mathematics, 1991) These moves for curriculum reform are supported by recent research into the potential benefits of student discussion for developing mathematical understanding (e.g. Goos, Galbraith & Renshaw, 1996) . However, research in this area has largely dealt with problem solving contexts, and has given scant attention to the possibilities for discussion that may arise when students try to make sense of explanations or examples in mathematical text. In fact, there has been little research on the use of mathematics textbooks in Australian classrooms, possibly because students so rarely read their textbooks for instructional purposes. Instead, students tend to use mathematics textbooks as a source of exercises, while it is the teacher who reads the textbook and then transmits the information to the class (Shield, 1991).

One explanation for this pattern of textbook use may stem from the differences between the language of mathematics and students' everyday language (MacGregor, 1989). The vocabulary, symbolism, syntax and concept density of mathematical text have all been identified as factors causing comprehension problems for students. For teachers, the solution to this problem does not lie in protecting students from difficult text by reading it for them and transmitting the information it contains. Instead, students need to gain experience in interpreting the explanations and examples in their textbooks—not only to help them come to terms with the rigorous formality of mathematical language, but also to encourage them to act as reviewers and critics of mathematical ideas produced by other people. The role of the text then changes, from that of a source of knowledge unchallenged in its authority, to a participant in students' learning conversations.

Lemke (1989) has described the process of using spoken language to make sense of instructional text as "making text talk". Students can make the text their own by talking their way to comprehension, elaborating and commenting on the text in multiple verbalisations which build connections between their own prior knowledge and language, and the formal language of textbooks. Since this process helps students become aware of their understanding, mediation of the text through peer or teacher-student discussion also has a metacognitive function.

This paper considers the question of how students come to understand mathematical text. From a metacognitive perspective, the assessment and development of understanding is seen in students' talk as they interrogate the text with peers, and in whole class discussion led by the teacher. However, the analysis also shows understanding in the making by identifying ways in which mathematical texts enter into different social contexts within the classroom. Sociocultural theories of learning view texts—together with other material and semiotic resources such as calculating devices, symbol systems, structures of reasoning and forms of discourse—as cultural tools which re-organise cognitive processes through their integration into human activity (Resnick, Pontecorvo & Säljö, 1997). From this perspective, then, understanding is scaffolded by social practices in which learners, teacher, and text are participants.

THE CLASSROOM STUDY

The research reported in this paper forms part of a larger study which investigated patterns of social interactions associated with metacognitive activity in senior secondary school classrooms (see Goos, 1998). While the study was carried out over a period of three years and involved eight teachers and their classes, this paper draws on data from a single Year 11 class in the third year of the study. Initial observations of this classroom revealed that the teacher regularly allowed time for students to study worked examples so that they would learn to find their way independently through mathematical text. The examples also introduced students to the formal reasoning involved in applying new concepts.

Subsequent observation identified three situations in which the students read and interpreted mathematical text: assigned reading, spontaneous reading, and reading peer-produced text. The teacher frequently *assigned* reading during lessons, allowing up to ten minutes for students to work through a piece of text themselves before reconvening for a whole class discussion. He did not expect the students to read in silence, but encouraged them to talk to peers to explicate the arguments and examples in the text. The teacher led the ensuing discussion by asking questions, such as *What happened there? So what's coming out of here? Is that right?*, in order to elicit detailed explanations of the text.

The second situation in which students read and interpreted mathematical text arose when they recognised their own lack of understanding of an aspect of the mathematics it presented, and joined with peers *spontaneously* to interrogate an example. Finally, there were occasions when students looked beyond the authority of the textbook and recognised the expertise of peers in producing exemplary texts, in the form of proofs or solutions to problems. *Student-produced texts* were interrogated for the same purpose as conventional textbooks—to understand someone else's thinking about a problem or mathematical idea.

An example of a lesson featuring spontaneous reading is presented next. The lesson comes from a unit of work which introduced students to fractals and chaos theory through the study of iterative processes. The unit was presented via a self-paced, teacher-prepared booklet, complete with examples and tasks, and students were expected to work together with minimal assistance from the teacher. All lessons in this three week program were videotaped, and portions of videotape were later transcribed for analysis.

Data Analysis and Coding

The analysis of the lesson transcript aims to identify features of students' collaborative metacognitive activity as they read mathematical text. The metacognitive function of students' dialogue is revealed through comprehension monitoring statements and explanations (Chi, Bassok, Lewis, Reimann & Glaser, 1989). Monitoring statements indicate that the students either understood, or failed to understand, the material presented in the text (noted in the transcript by the symbols *CM+* and *CM–*). Explanations are classified into one of four categories: (1) refine or expand the conditions of an action, (2) explicate or infer additional consequences of an action, (3) impose a goal or purpose for an action, and (4) give meaning to a set of quantitative expressions. (Only the third and fourth categories of explanation are illustrated in the sample transcript, annotated as *Exp-goal* and *Exp-quant* respectively.) Collaborative interaction is identified by sequences of conversational Moves which distribute a sentence between speakers, whether symmetrically, in the form of *collaborative completions*, or asymmetrically, as *instalment contributions* and partner *acknowledgments* (Clarke & Schaefer, 1989).

The Cantor Set

After the class had spent several lessons working on a set of activities related to the Koch Snowflake, the teacher asked students to read through the next example of a fractal, the Cantor Set. The text they were to read is reproduced in Figure 1. Note the error marked with a * : the common ratio is $r=2/3$, not $1/3$. This error was detected by Alex and Dylan, the students who are the subject of the present analysis.

When the teacher reconvened the class, he asked questions to elicit students' elaborations of the table showing Levels and Lengths of Section Removed, and then moved rapidly through the subsequent part of the example in which the sum of the lengths removed is found (Lines 1 to 4 in Figure 1). (T, A, and D identify the teacher, Alex and Dylan; R and L refer to other students in the class.)

1.	T:	OK, finding the sum of these terms because we are adding up these bits, are we not? Finding the total space removed? We can see that's a third plus two times a third squared, plus four times a third ... that's what we found? (Students murmur assent) OK, what have I done on the next line? Just turned that into powers of two. Next line's taken a third outside of it ... Yep? (Students focus intently on the text, nod)
2.	R:	Factorised it.
3.	T:	With a bit of sleight of hand we get two, we get a third, sorry we get one, two thirds, two thirds squared, two thirds to the third, two thirds to the *n* minus one—what does that represent? Sum of ...?
4.	L:	A GP.
5.	A:	(softly) Ooh I'd better learn up on that.
6.	T:	We want to sum that to infinity?
7.	D:	(softly, to Alex) What?
8.	A:	(softly, to Dylan) All this. I'd better go through that.
9.	T:	We want to sum that to infinity, find out what it adds up to totally, yep?
10.	D:	(hesitantly, to himself) Yeaahh.

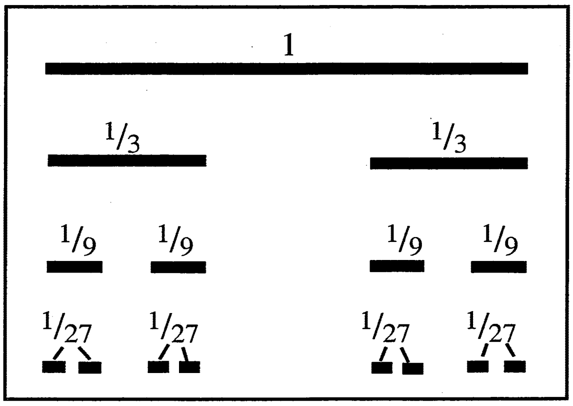
Alex and Dylan's uncertainty is clearly captured in their whispered asides to each other: they knew they had not understood the teacher's *sleight of hand* in moving through the steps labelled as Lines 1 to 4 in the example, since they had not reached this point in their initial reading. The teacher then asked the students to test their understanding of the construction of this fractal by finding how much space is removed from the Middle *Fifths* Cantor Set.

Almost immediately, Alex recognised that the answer would be the same as that for the original example, and then suggested—incorrectly—that they need only replace all instances of $1/3$ in the Middle Thirds example with $1/5$. Nevertheless, Dylan's uncertainty prompted him to re-examine the Middle Thirds example when he reached the point of reproducing Lines 1 to 4 in Figure 1. The analysis begins with Dylan's announcement that he wanted to go through the part of the example which applied the S_∞ formula, indicating a lack of understanding that he wished to remedy:

22.	D:	(makes a decision) I'm going to go through this thing (the example) and see *why*. *(CM–)*

Figure 1.
Middle Thirds Cantor Set

One of the most challenging fractals, in the sense of a common understanding of geometry, is the Cantor Set, yet it is the simplest. It is constructed by starting with an interval of length 1 and removing the middle third, leaving the two remaining end intervals. As you might have guessed, this is the beginning of an iterative process, so the next step is to remove the middle third of the two remaining intervals. This can of course be repeated infinitely many times. The full name of this construction is the Cantor middle-thirds set, which is represented diagrammatically below:



Each time the process of removing the middle third is carried out the amount of space between elements of the set increases. It might be interesting to find out the total amount of space removed as it would appear that there might be nothing left after the final iteration.

After the first iteration $^1/_3$ of the interval has been removed. After the second iteration 2 sections of length $(^1/_3)^2$ have been removed and after the third interval 4 sections of length $(^1/_3)^3$ are deleted. This gives us the following pattern:

Level	Length of Section Removed
0	0
1	$^1/_3$
2	$2\text{x}(^1/_3)^2$
3	$4\text{x}(^1/_3)^3$
n	$2n\text{-}1\text{x}(^1/_3)^n$

Finding the sum of these terms:

$S_\infty = {}^1/_3 + 2(^1/_3)^2 + 4(^1/_3)^3 + 8(^1/_3)^4 + \ldots\ldots$ *Line 1*

$= {}^1/_3 + 2^1(^1/_3)^2 + 2^2(^1/_3)^3 + 2^3(^1/_3)^4 + \ldots\ldots + 2^{n-1}(^1/_3)^n$ *Line 2*

$= {}^1/_3[1 + 2^1(^1/_3)^1 + 2^2(^1/_3)^2 + 2^3(^1/_3)^3 + \ldots\ldots + 2^{n-1}(^1/_3)^{n-1}]$ *Line 3*

$= {}^1/_3[1 + (^2/_3)^1 + (^2/_3)^2 + (^2/_3)^3 + \ldots\ldots + (^2/_3)^{n-1}]$ *Line 4*

Th^ section in the large set of brackets represents a converging GP with a = 1 and $r = {}^1/_3$ *. Using the formula for the sum to infinity of a converging GP:

$S_\infty = {}^a/(1 - r) = {}^1/(1 - {}^2/_3) = {}^1/(^1/_3) = 3$

Thus the sum of all the lengths removed is 1/3 x 3 = 1, which is the length of the original section. This is quite a remarkable occurrence: no wonder it is said that the Cantor Set is made up of dust. In fact, the only points remaining are the end points of each interval.

The remainder of the transcript is divided into segments corresponding to the boys' stepwise interrogation of Lines 1 to 4 in Figure 1.

Segment 1—Form Powers of 2 (Lines 1 and 2)

The only element of this segment of transcript which dealt substantively with the example is found in Move 30, where Dylan recognised that Line 1 was obtained by writing out the sum of the lengths of each section removed.

27. A: (Leaning earnestly over his booklet) What's this bit up here? Sum to infinity. Maybe that bit'd help. That little (thumps place on page where the formula appears) S squiggle. *(CM–)*
28. D: S infinity.
29. A: Yeah.
30. D: (gesturing to table of Levels and Lengths Removed) So it's all these added up. So a third ... plus two four eight etcetera— *(Exp-quant)*
31. A: (simultaneously)—plus ... OK, yeah, etcetera. *(CM+)*
32. D: A third ... (pointing to his page) these, now this is, they've got term numbers here now. Two to power one, two to power two ... two to power three, yeah that's making sense. *(CM+)*

Both boys confirmed their understanding of Lines 1 and 2 (in which 2, 4, 8 are simply written as $2^1, 2^2, 2^3$), by commenting that the example was now *making sense*.

Segment 2—Factorisation (Line 2 → Line 3)

This segment of the transcript again begins with a monitoring statement indicating comprehension failure, *what's this one here?*, initiating a series of explanations which gave meaning to the connection between these two lines of working.

37. A: So this one, what's this one here? *(CM–)*
38. D: Now they've bunged them all back, this one.
39. A: But why is it, from there to there, what happens here? *(CM–)*
40. D: A third, one. Why have they got *one*? Is that ...? One plus ... A third ... that's because you— *(CM–)*
41. A: Ahh! They've brought the, they've moved the powers back one (writing something on the example). *(Exp-quant)*
42. D: (looking at example) Huh? *(CM–)*
43. A: (taps Dylan's shoulder) Look. (Dylan looks at Alex's copy of the example) The common, common thingo, is a third, right? *(Exp-quant)*
44. D: Yeah.
45. A: So they've put that outside the brackets. So they've just done that, made that one (writes on the example), made that two. See that? They've got the third there— *(Exp-quant)*
46. D: Yeah.
47. A: —times one, so that is now that third there. (glances briefly at Dylan) *(Exp-quant)*
48. D: That's that (points to example in Alex's booklet) ...
49. A: Plus, that's a third, that now means a third, plus *two* times a third, *squared* ... *(Exp-quant)*
50. D: Ahh. *(CM+)*
51. A: Because it's a third *times* that. (Dylan nods) Right? And so on until you get, plus a third times ... (expression of sudden understanding on his face) yeah see, see now it's to the *n* minus one at the top there. *(Exp-quant)*
52. D: Yeah. (now looking at his own copy of example) *(CM+)*
53. A: Right. That's making sense. *(CM+)*
54. D: Sort of! *(CM–)*

Alex signalled that he understood how Line 3 was obtained by explaining that *they've moved the powers back one*. However, this cryptic statement was not a clear enough explanation for Dylan *(Huh?)*. Alex then delivered an extended elaboration of his initial explanation, as instalment contributions in Moves 43, 45, 47, 49 and 51. Two issues needed clarification: why the powers of $1/3$ had been reduced by one, and the origin of the 1 inside the large bracket. Both were explained by Alex pointing out that the common factor of $1/3$ (*common thingo*) had been taken outside the brackets. Despite acknowledging each instalment, and even indicating some understanding, Dylan did not share Alex's confidence that the example was now making sense (Move 54).

A striking feature of this segment of the transcript is the way in which Alex directed Dylan's attention to salient features of the example, and checked Dylan's understanding of each element of the explanation as it was delivered. For example, Alex ensured that Dylan was ready to pay attention by tapping him on the shoulder and indicating that they should both look at Alex's copy of the example in the Chaos booklet. He also glanced at Dylan to search for signs of understanding, and maintained their shared focus by punctuating his explanation with questions such as *Right?*, and *See that?*.

Segment 3—Express as a Geometric Series (Line 3 → Line 4)

Satisfied that he could justify the first three lines of working, Alex moved on to Line 4, in which the index law $a^nb^n = (ab)^n$ was applied. In this segment of the dialogue, the boys made a significant conceptual advance by recognising that the goal was to obtain an expression in the form of a geometric series.

55. A: (still staring at his booklet, Line 4) And then all of a sudden they've decided they've got two's as a common factor in there ... *(CM–)*
56. D: Yeah, well that's right.
57. A: And the power— Oh they're going so they get the powers the same! So you can put two in there. *(Exp-goal)* (Pause, Dylan looking at his own booklet) See that? (Alex indicates his copy of the example, then points to the place in Dylan's) See that? I mean they put that—
58. D: They get each, denominator— *(Exp-goal)*
59. A: They did this section here so, yeah, this power there— *(Exp-goal)*
60. D: That's not a power!
61. A: (still looking at own booklet) —yeah, so that power there is the same as that there. Three three, two two.
62. D: So they [inaudible] that—
63. A: So they can put the two into the, thing inside.
66. D: (to himself, looking at own booklet) Two thirds, two thirds, two thirds.
67. A: So two thirds to the n minus one.

Again it was Alex who initiated the explaining process, by identifying the reason for the algebraic manipulations of Lines 1 to 4: *they're going so they get the powers the same*. Once more, too, Dylan hesitated, and did not respond to Alex's repeated queries of *See that?* Perhaps sensing his partner's uncertainty, Alex started to elaborate further; but Dylan immediately joined in and the students constructed the justification seen in Moves 57 to 63 as a series of collaboratively completed sentences.

Segment 4—Error Detection

While the final segment of the transcript does not deal explicitly with the steps in Lines 1 to 4, it does reveal the outcome of the boys' purposeful interrogation of this section of the example. Dylan continued by reading aloud from the text.

68. D: (reading) "The section in the large set of brackets represents a converging GP with a is equal to one and r is equal to a third."
69. A: But isn't r equal to *two* thirds? *(CM–)* (Pause) Yeah it is (confident, taps Dylan's shoulder), they've just done it wrong because you look down there ... look ... it says two thirds there (showing Dylan the place in a subsequent step), so it is two thirds.
70. D: Ah. (turns to teacher, sitting behind) Mr G, is this a mistake? (Holds up booklet for teacher to read)
71. A: (to teacher) Is that meant to be two thirds?

72. T: (reads example) That should be two thirds.
73. A: (pleased, vindicated) Right, good! OK! Sum to infinity thing, OK. *(CM+)*

Alex's query, *But isn't r equal to two thirds?*, expressed his incomprehension of the text as it stood, and brought to light the error which had remained unnoticed by all until now. Although the boys called on the teacher to verify their claim, they did so with a degree of confidence that indicates a willingness to actively question, rather than passively accept, the mathematical ideas presented in the text.

Despite their success in making sense of these four lines of the Middle Thirds Cantor Set example, Alex and Dylan persisted with their over-generalised strategy of replacing $^1/_5$ with $^1/_3$ for attacking the Middle Fifths problem, as they still overlooked the need to draw a fresh diagram and construct from first principles a table showing how much space was removed with each iteration. When Dylan completed his calculations and announced that his answer was $^1/_3$, both boys were puzzled at the apparent contradiction between the calculated and expected results. However, this impasse was only recognised as the lesson ended, and they had to wait until the following lesson for its resolution.

Postscript

In the next lesson when the teacher asked for a volunteer to outline the solution process for the Middle Fifths problem, it became clear that many students had made the same error as Alex and Dylan. Rhys was one of the few students who had produced the correct solution, and he offered a detailed, but rapid, justification of the steps which led to the pattern $^1/_5 \times 2^{n-1} \times (^2/_5)^{n-1}$ for the length of section removed. As soon as the teacher ended this lesson phase and instructed the class to continue with a new task, Alex and Dylan moved to Rhys's desk and began examining his solution, which had now taken on the authoritative status of a worked example. Unlike the situation with textbook examples, the author himself (Rhys) was available to make his own text talk, enabling the readers to compare their interpretation with the author's intentions. Alex and Dylan eventually recognised the cause of the different patterns for the Middle Thirds and Middle Fifths Sets; however, the most convincing evidence of their understanding appeared when they went beyond these specific examples and attempted to find a general form for the algebraic patterns that emerge from the construction of any Cantor Set.

DISCUSSION AND CONCLUSIONS

From the teacher's perspective, the lessons in the unit of work on Chaos Theory followed a repeated pattern of assigned reading followed by whole class discussion. However, the students had their own agendas which were organised around their desire to understand the mathematics presented in the text, and they frequently delayed complying with the teacher's instructions to start work on a new task until they were satisfied that they understood the ideas introduced in the preceding section. As well as initiating this spontaneous reading, students also interrogated peer-produced text whose authority had been validated through whole class discussion.

How did the text come to be understood? From a metacognitive perspective, this question is answered by considering patterns of comprehension monitoring and collaborative explanations which re-interpreted an example and highlighted important information. Comprehension failure was usually followed by explanations which initiated the inferencing process necessary for restoring understanding, so that repeating cycles of the form *Comprehension failure (CM–)→Explanation →Comprehension restored (CM+)* became apparent as students interrogated each step in an example. This activity proceeded through collaboratively completed sentences or explanations delivered in instalments with partner acknowledgments, punctuated by verbal and non-verbal cues to listen to what was about to be said.

Although the students' explanations helped them to understand the text they were reading, comprehension was not purely an "in-the-head" process—understanding was scaffolded by social interactions involving students, teacher, and the text itself as participants. In making the text talk, students initially made much use of informal language; for example, Alex at first verbalised the symbol S_∞ as *S squiggle*, while Dylan exclaimed *Now they've bunged them all back* when he noticed that the powers of $1/3$ had diminished by one. While the students' colloquial language served an important purpose in helping them make sense of the ideas in the text as they worked with their peers, during whole class discussion the teacher explicitly guided them towards more formal articulation of these ideas by insisting that every action taken in the worked examples was fully explicated.

Current moves for mathematics education reform promote student interaction and discussion and criticise over-dependence upon the textbook for explanations of concepts and procedures. Yet, as the analysis presented in this paper has demonstrated, mathematical text can mediate social interaction between students in a way that transforms their relationship with mathematical knowledge. Rather than treating it as a pre-packaged object containing facts to be absorbed, the students interrogated, critiqued and appropriated the text by speaking its patterns in their own language. Their actions show that mathematics texts may still have a place in inquiry based classrooms—not as a source of exercises, but as a stimulus for critical engagement with mathematical ideas.

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