A research-based framework for assessing early multiplication and division strategies

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This paper describes the integration of multiplication and division strategies within a research-based Learning Framework in Number (NSW Department of Education and Training, 1998). The framework, consisting of six levels of multiplication and division knowledge is described in order of increasing sophistication of modelling, counting, grouping and sharing processes. Assessment tasks show a progression from initial grouping and counting to abstract composite units, to repeated addition and subtraction and to multiplication and division as operations. The framework, which is integral to the Count Me In Too Project is currently being implemented in years K-2 in NSW schools.

The purpose of this paper is to describe how long term research on children's development of multiplication and division has been translated into a Learning Framework in Number in order to assist professionals in assessing and improving children's number knowledge. The paper describes the theoretical framework and research basis which includes longitudinal research of 4-9 year olds development of multiplication and division though clinical interviews and teaching experiments. The main focus of the paper is to describe explicitly, the links between levels of development and key assessment tasks so as to distinguish how we can promote the development of increasingly sophisticated multiplicative strategies in young children.

Background to the Framework

Recent studies show that young children can develop multiplication and division concepts in the first years of schooling and this highlights a problem that teaching practices are not necessarily focussed on children's potential mathematical development (Carpenter, Ansell, Franke, Fennema, & Wiesbeck, 1993; Clark & Kamii, 1995, Kouba, 1989; Hunting & Davis, 1991; Mulligan & Mitchelmore, 1997). There is also growing evidence that once children reach the primary grades they are unable to solve problems involving multiplication and division or apply multiplicative number facts with meaning. In the upper grades, students find difficulty in using multiplicative reasoning in a range of contexts and integrating their understanding of rational number with multiplication and division (Behr, Harel, Post, & Lesh, 1994; Bell, Greer, Grimison & Mangan, 1989; Confrey & Smith, 1995).

Multiplicative reasoning is essential in the development of concepts and processes such as ratio and proportion, area and volume, probability and data analysis. It is also clear that failure to develop multiplicative structures in the early years impedes the general mathematical development of students into the secondary school, for example, in using algebra, functions and graphs. A concomitant problem is that multiplicative concepts are often not well understood or well taught by teachers at primary and secondary level (Graeber, Tirosh & Glover, 1988; Simon, 1993). It appears that difficulties faced by older students can be attributed, at least in part, to the lack of development of an equal-grouping structure in early concept formation (Mulligan & Mitchelmore, 1997; Steffe, 1994).

An Overview of Research on Multiplication and Division

Studies investigating multiplication and division processes with younger children have identified the development of sound problem-solving strategies from an early age and the importance of modelling and representation in this development (Anghileri, 1989; Carpenter, Ansell, Franke, Fennema & Weisbeck, 1993; Clark & Kamii, 1996; Kouba, l989; Mulligan & Mitchelmore, 1997; Steffe, 1994). Recent research on multiplicative reasoning has looked to the early development of multiplication and division and fraction concepts, through essential processes such as counting, partitioning, grouping, unitizing and 'splitting' (Confrey, 1994; Hunting & Davis, 1991; Lamon, 1996; Steffe, 1994).

In longitudinal analyses of young children's intuitive models for multiplication and division problems Mulligan & Mitchelmore (1997) found that the intuitive model employed to solve a particular problem did not necessarily reflect any specific problem feature but rather the mathematical structure that the student was able to impose on it. Students acquired increasingly sophisticated strategies based on an equal-groups structure and calculation strategies that reflected this. Counting strategies were integrated into repeated addition and subtraction processes and then generalised as the binary operations of multiplication and division. Strategies used with concrete and sensory models were internalised and replicated at an abstract level with increasing sophistication. Although multiplicative reasoning does not usually emerge in instructional programs until the second or third grade the Count Me In Too Project assesses the development of these processes in an attempt to seek out new ways of addressing the difficulties experienced in learning and to formulate more valid assessment techniques than traditional multiple choice tests.

Classroom-based studies on teaching and learning multiplication and division have also investigated the ways in which children devise and represent related problem situations and solve computations (Boero, Ferrari & Ferrero; Carpenter et al. 1993; Lampert, 1990; Mulligan & Mitchelmore, 1996; Murray, Olivier & Human, 1992; Nesher,1988). Teaching approaches based on the the Count Me In Too Project aim to encourage closer links between concrete and abstract thinking in order to promote increasingly sophisticated strategies based on the levels of described later in Table 2.

The development of composite structure; A developmental framework assessing the growth of multiplication and division processes must be based on the acquisition of an equal-grouping (composite) structure which is at the heart of multiplicative reasoning. A composite whole is a collection or group of individual items that must be viewed as one thing. For example, a child must view three items as "one three" in order for the unit "three" to be a countable unit. For a true understanding of multiplication and division the child needs eventually to co-ordinate groups of equal-sized groups and recognise the overall pattern i.e. composites of composites, e.g."three sixes". Steffe (1994) describes this as a premultiplying scheme, "For a situation to be established as multiplicative, it is necessary at least to co-ordinate two composite units in such a way that one of the composite units is distributed over elements of the other composite unit" (p.19). Other theorists emphasise the importance of this structure calling it unitizing (Lamon, 1996), and re-initialising (Confrey, 1994). As well, multiplication and division may require that quantities or the numerical referents are changed or transformed as a result of the process. The quantity that is the product is a different type of quantity to the two like or unlike quantities that have been multiplied, e.g. where combinations of t-shirts and jeans produce "outfits".

Once the initial elements are developed and consolidated with repeated addition or repeated subtraction and sharing models, multiplicative reasoning must extend beyond these to a point where the commutativity of multiplication is recognised and the inverse relationship between multiplication and division is applied. The development of multiplication and division as inverse processes forms the basis of a developmental model of composite structure. The acquisition of multiplication and division as binary operations relies on the child's ability not only to develop composite structure and commutativity but also to recognise the relationship m x n where m is the composite unit "operated upon" n times. This is quite different to a repeated addition notion of multiplication which is commonly used in teaching practice.

Children may use identical or similar strategies for solving both multiplication and division tasks except that in division, the child will form and count composite units from an existing quantity. Interestingly, it has been found that division is not necessarily more difficult than multiplication, and in many situations, division situations may be easier than multiplication. For example, it may be easier for a child to share counters into equal groups and count the number of groups rather then keeping track of a larger number of composite groups for multiplication. Teaching children to share and group small numbers into equal parts can facilitate the development of multiplication and division strategies ie. non-count by ones strategies (Mulligan & Mitchelmore, 1996).

The Learning Framework in Number

Five key components shown in Table 1 have been developed as key aspects of early number learning. Together, parts A, B, C, D, and E form an interrelated framework (see Wright, 1998). While early arithmetical strategies form the central structure of the framework other aspects such as multiplication and division should not be seen as compartmentalised or in developed in isolation.

Children's early multiplication and division knowledge is based fundamentally, on the development of counting sequences and arithmetical strategies (Part A), along with skills of combining, partitioning and patterning (Part E). The concepts of multiplication and division are not only interrelated, but are closely linked with addition and subtraction strategies, and early fraction learning. For a true understanding of multiplication and division the child needs to eventually co-ordinate groups of equal groups and recognise the overall pattern ie. composites of composites eg. "three sixes".

Table 1: Aspects of Early Number Knowledge

A1	*Early Arithmetical Strategies*
0	Emergent Counting
1	Perceptual Counting
2	Figurative Counting
3	Counting On
4	Facile Number Sequence
B1	*Forward Number Word Sequence*
B2	Backward Number Word Sequence
B3	Numeral Identification
C1	*Early Multiplication and Division Strategies*
1	Initial Grouping and Perceptual Counting
2	Intermediate Grouping and Counting
3	Abstract Composite Units
4	Repeated Addition and Subtraction
5	Multiplication and Division as Operations
C2	*Fraction Knowledge*
D	*Base-Ten Arithmetical Strategies and using "Five"*
D1	Base-ten Strategies
D2	Quinary-based Strategies
E	*Early Arithmetical Procedures*
E1	Combining and Partitioning
E2	Spatial Patterns and Subitising
E3	Temporal Sequences
E4	Finger Patterns

Table 1 places multiplication and division strategies in the central section of the framework. It should be noted that children who have progressed only to Figurative Counting (A1.2) in early arithmetical strategies are unlikely to construct composite units as counting units (iterable units). However, children who are able to partition using five and ten as a base, including quinary-based D1 - D2, make "five", partitioning "five", and counting in fives using equal groups, are likely to be developing composite units.

The development of combining, partitioning and patterning strategies included in Part E as well as other early arithmetical strategies of addition and subtraction can be effectively linked with the development of multiplication and division strategies. Various aspects of multiplication and division can be seen as integrated or overlapping with these strategies. Thus, two or more different situations involving multiplication and division may elicit the same arithmetical strategies.

The Development of Multiplication and Division Strategies

A framework consisting of six levels of multiplication and division knowledge is described in order of increasing sophistication of modelling, counting, grouping and sharing processes. At the same time, the development of multiplication and division as operations relies on the child's ability to co-ordinate composites ie. use groups of equal groups as single entities. Levels of this framework are based on longitudinal analyses of children's development of multiplicative strategies from Grade 2 through 3 (see Mulligan & Mitchelmore, 1997).

Six levels are described below in Table 2 giving descriptions of the main types of strategies. It is noted that children may demonstrate a combination of these strategies depending on the size and type of the numbers used. For example, children find counting and grouping in 2's, 5's and 10's easier than 3's or 4's. Generally, children will progress from *initial composite counting*, to *abstract composites,* to *multiplication and division operations,* and to *known facts.*

Table 2 Development of Multiplication and Division Strategies

C1.1 Initial Grouping and Perceptual Counting: models or shares by dealing in equal groups but they do not see the groups as composite units; count each item by ones *(perceptual).*

C1.2 Intermediate Composite Units: models equal groups and counts using rhythmic, skip or double counting; counts by ones the number of equal groups and the number of items in each group at the same time only if the items are visible.

C1.3. Abstract Composite Units: models and counts without visible items ie the child can calculate composites when they are screened, where they no longer rely on counting by ones. The child may not see the overall pattern of composites such as "3, 4 times".

C1.4 Repeated Addition and Repeated Subtraction:: co-ordinates composite units in repeated addition and subtraction. Uses a composite unit a specific number of times as a unit e.g. 3 + 3+ 3 +3; may not fully co-ordinate two composite units.

C1.5 Multiplication and Division as Operations:
Two composite units are co-ordinated abstractly eg. "3 groups of 4 makes 12"; "3 by 4" as an array; "3...4 times as many, "12 into 3 groups...that's four; "3, four times".

C1.6 Known Multiplication and Division Fact Strategies: recalls or derives easily, known multiplication and division facts; uses multiplication and division as an inverse relationship.

Assessment tasks; Several critical assessment tasks are described explicitly in Appendix A. These tasks form a basis for ascertaining the child's level of development of composite structure. Appendix A shows a progression from counting strategies based on multiples to the formation of composites without visible models. As for other arithmetical tasks (Table 1, A1), the assessment focusses on the child's ability to use strategies that are increasingly more sophisticated and assesses the child's ability to use composite units abstractly, i.e. calculate groups and the number of groups when items are screened. For multiplication and divison tasks there is a distinct difference in difficulty between partition and quotition tasks (Appendix A , C4.2 and C4.3) and the importance of an abstract double counting strategy (C4.4) when items are screened.

Discussion

The development of efficient counting (non-count by ones, skip and double counting) and composite units are integral to developing composite structure. Co-ordinating composite units, e.g. " three threes as a unit of 9" depends on the ability to move beyond counting based on a unitary notion and to use a pattern of multiples as a double count ("1, 2, 3 (one), 4, 5, 6 (two)" etc.) mentally. While the development of direct counting and visual modelling precedes development of abstract composite structure there exists a complex interrelationship between counting and composite structure at the abstract level. The use of skip and double counting procedures gives rise to more efficient processes that take advantage of the equal-grouping structure where repeated addition (or subtraction) is generalised as an operation.

The development of repeated addition or repeated subtraction at level C1.4 (see Table 2) does not constitute a full conceptual understanding of multiplication or division. The next level C1.5 distinguishes the development of multiplication and the related division process as the distribution of a composite unit across elements of another composite unit e.g. generalising the structure of composites, for example, as "six, three times as 6 x 3 = 18". Critical to developing this relational understanding of multiplication is the ability to see multiplication and division in an inverse relationship and to explain commutativity such as 6 x 9 = 9 x 6. Children who are able to simply recall multiplication and division number facts without being able to explain and represent the composite structure are not yet functioning at this level.

Implications

The assessment and development of multiplication and division strategies in the early years of schooling requires professionals to integrate key aspects of developing composite structure within number learning generally. Although grouping and sharing processes may on the surface, form part of "traditional" practice, the systematic and explicit nature of the framework allows professional to gain further insight into matching learning experiences with the child's potentialities. Anecdotal evidence of the initial implementation phase has indicated that children in years K-2 are capable of using multiplication and division strategies effectively across a range of learning situations. The implementation and extension of the framework into more complex problem-solving tasks will have implications for curriculum reform and teaching practices in the early years.

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Appendix A : Assessment Tasks for Early Multiplication and Division

C1.Tasks Assessing Number Word Sequence in Multiples

The purpose of these tasks is to assess the child's facility with the forward number word sequence in multiples of 2 and 3, i.e. skip counting by twos and threes. This is a basic task where the teacher can discriminate between the child's facility with skip counting produced verbally, by co-ordinating skip counting with visible items, and by co-ordinating skip counting with a numeral track.

C1.1 Counting forward number word sequence in multiples of 2 and 3

The teacher asks the child to begin counting in twos until directed to stop beginning with the number 2. When the child has reached 20 the teacher directs them to stop. The task is repeated with directions to count in threes but it is anticipated that this is more difficult. When the child has reached 12 the teacher directs them to stop.

C1.2 Counting visible items in a row by multiples of 2 and 3

The child is presented with a row of 20 counters and is directed to count them in twos. The teacher observes whether the child co-ordinates correctly the number word sequence of twos with the items. The task is repeated with directions to count a row of 12 counters in threes.

C1.3 Counting by multiples of 2 and 3 using a numeral track

The child is presented with a numeral track 1-10 and is directed to count forwards in twos by correctly co-ordinating (by touching) the numerals with a verbal count. The task is repeated for counting by threes. An extension of this task is to ascertain whether the child can identify hidden numerals in a sequence of twos or threes. Several numerals in multiple patterns of twos or threes are covered. The child has to identify the hidden numerals.

C2 Tasks Assessing Equal Grouping and Sharing with Visible Items

The purpose of these tasks is to establish whether the child can construct **initial composite** units by equal grouping in multiplication and division situations. There are three basic tasks where the child makes specified groups by grouping and sharing. The teacher observes whether the child is unable to coordinate the number of equal groups and the number in each group at the same time.

C2.1 Composite units task by equal grouping (multiplication) using visible items

The child is asked to make specified equal-sized groups e.g. make 4 groups with 3 in each group. This is done using one to many matching where the number of groups is already modelled. The child places 3 lego people on each lego train or 3 counters on each plate where the lego train or the plate represents the number of groups. The child is asked to calculate the total number of items.

C2.2 Composite units task by sharing (partition division) using visible items

The child is asked to share a specified number of items into a specified number of groups and calculate the number of items in each group e.g. share (equally) 12 items among 3 children, how many do they have each? This is done using one to many matching where the number of groups is already modelled e.g. items to represent children.

C2.3 Composite units task by grouping (quotition division) using visible items

The child is asked to make specified equal-sized groups where the number of groups is unknown and to calculate the number of groups e.g. 12 items are shared among the children and they get 4 items each. How many children are there? The teacher observes how the child forms groups of 4 items until the 12 items are exhausted.

C3 Tasks Assessing Use of Composite Units Involving Partially Screened Items

The purpose of these tasks is to establish whether the child can construct **intermediate composite** units by equal grouping in multiplication and division situations. The basic tasks (C2), where the child makes specified groups by grouping and sharing using visible items and uses referents for the groups, are repeated but this time the items are partially screened. The teacher observes whether the child is unable to coordinate the number of equal groups and the number in each group at the same time.

C3.1 Composite units task by equal grouping (multiplication) partially screened

The child is asked to make specified equal-sized groups e.g. there are 4 containers with 3 items in each container. e.g. The child places 3 items in each of 4 containers (opaque). The child is asked to calculate the total number of items in the containers where the items cannot be seen or counted.

C3.2 Composite units task by sharing (partition division) partially screened

The child is asked to share a specified number of items into a specified number of groups and calculate the number of items in each container e.g. share (equally) 12 items into each of 3 containers, how many items in each container? The child is required to keep track of the number of items in each container where the items cannot be see or recounted.

C3.3 Composite units task by grouping(quotition division) partially screened

The child is asked to make specified equal-sized groups where the number of groups is unknown and to calculate number of groups e.g. 12 items are shared into containers where the child knows there are four in each container. The teacher observes how the child forms groups of 4 items until the 12 items are exhausted. The child is required to keep track of the number of counters in each container where the items cannot be see or recounted.

C4 Tasks Assessing Use of Abstract Composite Units

The purpose of these tasks is to establish whether the child can construct **abstract composite** units by equal grouping in multiplication and division situations. The basic tasks (C2) are repeated without any concrete or visible items. The child is required to coordinate the number of equal groups and the number in each group at the same time. An abstract double counting task (C4.4) naturally evolves from the C4.3 task with grouping (quotition division) and is important for assessing coordination of composite units.

C4.1 Composite units task by equal grouping (multiplication)

The child is asked to calculate the total number of items in an equal grouping problem where the items cannot be seen or counted in an equal grouping problem e.g. there are 4 groups with 3 items in each group. How many are there altogether?

C4.2 Composite units task by sharing (partition division)

The child is asked to calculate the number of items in each group e.g. share (equally) 12 items into 3 groups, how many in each group? The child is required to keep track, mentally, of the number of items in each group where the items cannot be see or counted.

C4.3 Composite units task by grouping (quotition division)

The child is asked and to calculate the number of groups where the number of groups is unknown e.g. 12 items are shared into groups where the child is told there are four items in each group. The child is required to keep track of the number of items in each group where the items cannot be see or counted

C4.4 Abstract double counting task by grouping (quotition division)

Repeat task C4.3. or ask similar division tasks e.g. there are 12 biscuits and the children are given 3 biscuits each. How many children are there? The teacher observes whether the child coordinates the number of groups and the number of items in each group simultaneously by asking the child to talk aloud as they solve the task. The teacher observes whether the child uses a referent such as fingers to keep track of the number of groups e.g. " 1, 2 , 3 (1)...4, 5, 6, (2), ...7, 8, 9,(3)... 10, 11, 12 (4). The task can be repeated using more difficult numerical grouping such as 5 groups of 3.

C5. Tasks Assessing Use of Composite Units in Arrays

Co-ordinating the equal-sized groups (composite units) as rows and columns and recognising the commutativity of multiplication e.g. 3 x 5 = 5 x 3, is an essential aspect of developing multiplication and division concepts.

C5.1 Calculate total number of items in a partially screened array

This task assesses the use of intermediate composite units where the visible composite units are used to count those that are screened. The child is presented with a 3 x 5 rectangular array of items with two rows screened. The child is told that there are two rows exactly the same as the other rows and asked to calculate the total number of items.

C5.2 Recognise commutativity of multiplication in an array

This task assesses the commutativity of multiplication where the composite units are visible. Present a rectangular array of 3 x 5 items fixed on a card e.g. stickers in rows and columns. Turn the array at right angles. Ask the child whether there are still the same number of items in total and to explain the grouping.

C5.3 Calculate total number of items in a screened array

This task assesses the use of **abstract composite units** where equal groups are visualised and counted when completely screened. A rectangular array of 3 x 5 items is screened. The child is required to visualise the screened array from a description of the number of rows and the number of items in each row and calculate the total number of items.