"I don't know if I'm doing it right or I'm doing it wrong!" Unresolved uncertainty in the collaborative learning of mathematics

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This paper reports on a study which investigated patterns of collaborative metacognitive activity in senior secondary school classrooms. Although peers working together on mathematical tasks may enjoy the metacognitive benefits of being able to monitor and regulate each other's thinking, collaboration does not guarantee that they will achieve a mathematically productive outcome. Analysis of a videotaped lesson transcript illustrates how metacognitive uncertainty, itself a trigger for collaboration, remained unresolved when students did not have the means of validating their solution.

The last decade has been marked by the emergence of new mathematics curriculum and policy documents which place increased emphasis on problem solving, mathematical reasoning, and communication, and promote peer interaction and discussion as a means of helping students to develop their understanding of mathematics (Australian Education Council, 1991; National Council for Teachers of Mathematics, 1989). However, our theoretical understanding of problem solving processes, and how students' mathematical thinking is shaped by their interaction with peers, is far from complete (e.g. Lester, 1994; Schoenfeld, 1992).

One aspect of mathematical thinking which deserves continued research attention concerns the role of metacognitive processes, that is, how students monitor and regulate their thinking while working on mathematical tasks. The importance of metacognition is now widely acknowledged, and many studies have investigated the metacognitive strategies which secondary school students use in problem solving. However, these studies have tended to focus on students working individually, in experimental settings, on tasks prescribed by the researcher (e.g. Fitzpatrick, 1994; Randhawa, 1994). Despite increasing research interest in the social and cultural aspects of mathematics learning (e.g. Brown et al., 1993), few studies have sought to examine the characteristics of collaborative metacognitive activity occurring when students work together in natural classroom settings, or the conditions under which such interaction leads to successful or unsuccessful problem solving outcomes.

The research discussed in this paper is part of a larger study which investigated patterns of classroom social interactions associated with metacognitive activity, and assumptions about teaching and learning mathematics underlying teachers' and students' actions. A major aim of the research was to suggest mechanisms through which peer interaction mediates metacognitive activity, and results reported previously have demonstrated that jointly transacted monitoring and regulation can help students to overcome the obstacles in their progress towards a successful solution (Goos & Geiger, 1995; Goos, 1997). Nevertheless, it would be misleading to claim that peer collaboration always achieves a mathematically productive outcome. The purpose of this paper is to identify circumstances in which collaboration may be metacognitively fruitless. One instance of metacognitive failure is examined in detail in order to illustrate some of the conditions responsible for students' lack of success.

Metacognitive Success and Failure

Frameworks for analysing task-oriented mathematical thinking typically identify phases or episodes representing distinctive types of problem solving behaviour, and describe the ideal characteristics of each episode (e.g. Artzt & Armour-Thomas, 1992; Schoenfeld, 1992). While these frameworks acknowledge the central role of metacognitive processes in keeping the solution process on track, they do not consider in detail the types of monitoring and regulatory activities that would be appropriate and expected at each stage of the solution. In particular, previous research in this area has not distinguished between the routine monitoring which serves merely to confirm that all is well, and the more controlled monitoring and regulatory processes triggered when students become aware of specific difficulties. It is helpful to think of these triggers as metacognitive "red flags", which signal the need for a pause or some backtracking while remedial action is taken. "Red flags" could be raised when students realise that they are making no progress, notice a calculation error, or recognise that their answer violates the problem conditions or does not make sense. Appropriate responses to each of these warning signals are shown in Figure 1.

"Red Flag"	**Congruent Response**
Lack of progress	Reassess appropriateness of the solution strategy. Decide whether to persist or abandon the strategy, identify useful information to be salvaged.
Detection of an error	Check and correct calculations.
Anomalous result	Check calculations. Reassess the solution strategy.

Figure 1. Metacognitive "red flags" and congruent responses

A further problem with existing analytical frameworks, and with research on metacognitive processes in mathematics generally, is the lack of explicit attention given to characterising different forms of metacognitive failure, other than to note that students did not exploit useful information, or that checking behaviour was absent. When students work individually or collaboratively on mathematics tasks, one could imagine that several different metacognitive scenarios might arise, as illustrated in Figure 2. While metacognitive *success* will occur if students recognise a "red flag" and take appropriate action to deal with the difficulty (or recognise that nothing is wrong and continue on the same solution path), less successful outcomes are likely in at least three other circumstances. First, students can be guilty of metacognitive *blindness* if they fail to notice that something is amiss, for example, by persisting with the wrong strategy or overlooking a calculation error. Second, students might commit metacognitive *vandalism* by taking inappropriate action to deal with an impasse, for example, by changing the problem to enable them to apply knowledge already available to them. Finally, the "red flag" itself may be spurious and represent a metacognitive *mirage* if students "see" difficulties which do not exist, and mistakenly abandon a useful strategy, amend calculations which are not in error, or reject correct answers.

A less obvious form of mirage materialises when students are unsure about what they "see" and, unable to make any judgment about the correctness of their strategy or answer, remain lost in the desert of uncertainty. The means by which such uncertainty is resolved have been studied by Clarke and his colleagues (e.g. Clarke & Helme, 1997), who have conducted an extensive study of mathematics learning as it occurs in legitimate classroom settings. Clarke and Helme (1997) argue that learning in social settings, such as classrooms, proceeds through the negotiation of meaning, and that the social goal of negotiation is to resolve uncertainty. They identify four forms which the process of resolution may take, basing the categorisation on the dominant authority to which students appeal in seeking resolution:

- prior experience;
- empirical evidence;
- a knowledgeable person; or
- a text.

It seems reasonable to conclude that students' inability to make metacognitive judgments may be due to lack of access to one or more of these forms of authority. Such a situation is illustrated in the remainder of the paper, which examines one case of unresolved uncertainty in a Year 11 mathematics classroom.

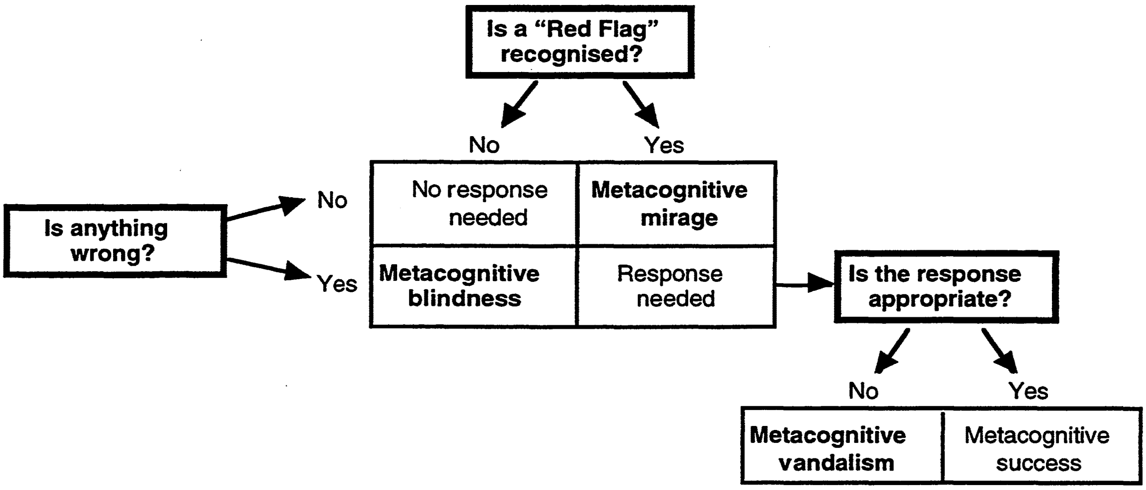

Figure 2. Metacognitive success and failure

The Classroom Study: Data Gathering and Analysis Methods

The study was carried out over a period of three years, and involved eight secondary school teachers and their Year 11 or Year 12 mathematics classes. Methods used to investigate students' individual and collaborative metacognitive activity included questionnaires, interviews, and classroom observation supplemented by audiotaping and videotaping. Target students were chosen for videotaping and interview on the basis of their metacognitive sophistication and preference for working collaboratively with peers, as judged from preliminary observation and responses to questionnaires (see Goos, 1995, for details of questionnaires). One lesson was observed each week, and target students were audiotaped and videotaped as they worked together and discussed their ideas in class. Portions of the videotapes were later transcribed for analysis. Semi-structured, individual interviews were carried out to probe and follow up some of the issues raised in the open ended metacognitive self-knowledge questionnaire. All interviews were audiotaped and transcribed.

Classroom videotape transcripts were first parsed into macroscopic episodes which categorised the group's problem solving behaviour as Reading, Understanding, Analysis, Exploration, Planning, Implementation, Verification, or Transition (episode characteristics are as described in Artzt and Armour-Thomas, 1992). A finer grained analysis of conversational turns was then carried out to identify metacognitive activity. Two types of metacognitive acts were coded: proposing a *New Idea* (recognising potentially useful information, mentioning an alternative approach); or making an *Assessment* of a particular aspect of the solution (such as one's understanding of the problem, the appropriateness or execution of a strategy, or the accuracy or sense of results) (Goos & Geiger, 1995). Analysis of the interview transcripts focussed on individual students' ability to recognise and respond to metacognitive "red flags".

Unresolved Uncertainty in Solving Combinations Problems

This paper draws on videotape and interview data from one Year 11 mathematics class in the third year of the study. The videotape transcript comes from a lesson in the early stages of a unit of work on combinatorics. Knowing that he would be unavoidably absent for this lesson, the teacher had set a series of problems which would give the students their first opportunity to apply their newly gained knowledge of combinations. These problems were contained in a teacher-prepared handout which also included explanations and worked examples, and served as the students' sole text for the topic. Four target students—Dylan, Alex, Sean, and Rhys—figure in the transcript. (Pseudonyms are used to preserve anonymity.) Figure 3 shows the first problem on which they worked (Question 19 in the problem set), together with model solutions.

1 9 How many selections of five cards can be made from a pack of 52 playing cards so that there are: **a** at least three aces? **b** three hearts? **c** at least one heart?

Solutions

a Possible hands could contain either three or four aces.

$$\therefore \text{ number of selections } = \left({}^{4}C_3 \times {}^{48}C_2\right) + \left({}^{4}C_4 \times {}^{48}C_1\right) = (4 \times 1128) + (1 \times 48) = 4512 + 48 = 4560$$

b The hand must contain three hearts and two non-hearts.

$$\therefore \text{ number of selections } = \left({}^{13}C_3 \times {}^{39}C_2\right) = 286 \times 741 = 211926$$

c The hand may contain either one, two, three, four, or five hearts. (A simpler method is to find the number of hands with no hearts and subtract this from the total number of five card hands.)

$$\therefore \text{ number of selections } = \left({}^{13}C_1 \times {}^{39}C_4\right) + \left({}^{13}C_2 \times {}^{39}C_3\right) + \left({}^{13}C_3 \times {}^{39}C_2\right) + \left({}^{13}C_4 \times {}^{39}C_1\right) + {}^{13}C_5 = (13 \times 82251) + (78 \times 9139) + (286 \times 741) + (715 \times 39) + 1287 = 2023203$$

Figure 3. Combinations problems

Episode Analysis

Analysis of the transcript is limited to the time in which the students jointly focussed on Question 19, before their solution paths diverged and social interactions became more fragmented and difficult to track. The following conventions were adopted in transcribing students' interactions: (a) conversational turns (referred to as Moves) are numbered sequentially; (b) students are identified via their initials; (c) non-verbal information from the videotape is included in parentheses; (d) the symbol (...) indicates that part of the transcript has been omitted; and (e) annotations indicating metacognitive acts are recorded in italics.

Episode 1—Reading/Understanding: After reading the stem to Question 19 and making the observation that there were ${}^{52}C_5$ hands in total, Dylan immediately recognised that this would give too large a number for part (a) of the question, which imposed the constraint of having at least three aces in the hand.

1. D: (...) So it's fifty-two C five. *(New Idea)* (No response from other students. They look at him, smiling, expressions of disbelief on their faces.) Sounds a lot, doesn't it. *(Assessment—strategy appropriateness)*

Nevertheless, all three boys used the ${}^{n}C_r$ buttons on their calculators to gain a feel for the problem by confirming that ${}^{52}C_5$ is indeed a large number (2 598 960).

6. D: (...) Anyway let's see what fifty-two C five is ... (uses calculator)
7. All: (in unison) Two million five hundred and ninety-eight thousand, nine hundred and sixty!

Episode 2—Analysis: Although they had identified the relevant information in the problem, the students struggled to formulate a strategy for taking account of the specified selection of at least three aces. Eventually, Alex and Dylan proposed that ${}^{52}C_2$ might represent the number of five card hands *without* three aces, foreshadowing an approach based on mutually exclusive operations and the addition principle. (Note that they still had not come to grips with the "at least" condition.)

15. A: (doubtfully) Yeah, but how do you work out these three aces?
16. D: No, you've got five cards, so it's only fifty-two, ah ... fifty-two C— *(New Idea)*
17. A: Ohh! Do C two, that's how many *won't* have— *(New Idea)*
18. D: Yeah, and you *got* to have—
19. A: (simultaneously) —a certain three cards.

Despite their initial enthusiasm for this strategy, it soon became apparent that the boys had no way of knowing whether or not they were on the right track.

24. D: Aah I don't know if I'm doing it right or I'm doing it wrong! *(Assessment—strategy appropriateness)*

Episode 3—Exploration: The students' uncertainty was manifest in the way they appealed to each other for assistance.

25. S: (to Alex) So what have you done for the first one?
26. A: (Unsure) I don't know. *(Assessment—understanding)* You go fifty-two ... (to Dylan) So how are you doing (a)? Dylan? Dylan? (Dylan is talking off-task to Rhys.) How are you doing (a)?
27. D: I don't know. *(Assessment—understanding)*
28. A: Are you doing fifty-two, C, two—
29. D: —two—
30. A: —and then subtract it?

Before long, the boys abandoned part (a) of the problem and acknowledged that they were stuck on Question 19 as a whole. Not yet willing to give up completely, they considered two potentially useful strategies for dealing with impasses such as the one they faced—working backwards from the answer (Moves 41 and 42), and looking at a similar problem (Move 45). Unfortunately, they were unable to take advantage of either strategy, since the teacher-prepared handout did not provide answers to the problems, and they overlooked a worked example in the text which might have provided some clues. While Alex continued to hunt for a helpful example in the text, Dylan moved on to Question 19 (b), and began hesitantly to reason out a strategy which would lead him to the correct answer (Move 44).

40. D: So how do you do it?
41. S: If we had an answer—an answer sheet— *(New Idea)*
42. D: Yeah, we could figure it out. *(New Idea)*
43. A: You could always think about it without the C rule. And go like, OK, for hearts you've got, however many choices, and, the next choice you've got however many choices, the next choice you've got ... *(New Idea)*
44. D: You got a quar—ter ... (hesitation, draws out this word) A quarter of fifty. *(New Idea)*
45. A: (Not listening to Dylan) Think about it the long way. Hey, is there an example somewhere? *(New Idea)* (Checks quickly through handout, overlooks Example 12.)

Episode 4—Implementation: In this segment of the transcript there is evidence that Dylan was beginning to develop a general understanding of how the choices of cards can be constrained. In the case of Question 19 (b), if a five card hand is to contain three hearts, then the hearts are selected from only one suit (a quarter of fifty-two cards), not the full pack.

46. D: What's a quarter of fifty? Twenty ... No, that's half of fifty.
47. S: Quarter of fifty, or fifty-two?
48. D: Quarter of fifty, because there's two Jacks (meaning Jokers). Oh hang on. Is it fifty-two including, Jokers? Or fifty-four?
49. A: (Still flicking through handout, searching for example.) No, fifty-four.
50. D: Fifty-four. Then it's a quarter of fifty-four.
51. A: (Unable to find an example) We could probably do it if we thought about the long way—
52. R: (to Dylan) That's something and a half. Twenty-five and a half.
53. D: (Uses his calculator; looks at Rhys) Thirteen. (Rhys and Dylan laugh.)
54. D: (writes) This is thirteen out of fifty-two ... is ... hearts. So what would you go? Would you go, thirteen ... C ... [inaudible]. *(New Idea)*

Transition (Moves 55-62): Having successfully, if somewhat laboriously, calculated the number of hearts in the pack, Dylan now became absorbed with completing Question 19 (b), and he worked in silence while his friends considered their next move.

55. S: So are you still trying to work out something for—?
56. A: No, I'm just going to leave that for now. And wait until *he* comes up with—

57. S: Leave Question 19 altogether?
58. A: Yeah, I don't know how to do it. *(Assessment—understanding)*

Episode 5—Implementation: The start of this episode is heralded by Dylan's triumphant announcement that he had worked out how to attack Question 19 (b). By this time, however, Alex and Sean were busily working on Question 20, and they did not acknowledge his breakthrough.

63. D: (to himself) So should we go ...? I know, I've figured it out! I've figured it out! *(Assessment—Understanding)* (Pause) Multiply that by ... what's the [inaudible]? It's thirteen take fifty-two. *(New Idea)*
64. R: Thirteen *take* fifty-two? *(Assessment—strategy execution)*
65. D: Sorry! Fifty-two take thirteen. Thirty-nine, yeah. (Quietly, to himself) Thirty-nine C two.
66. A: (Reading Question 20) How many committees of five ...?
67. D: (to himself, using calculator) Two hundred and eighty-six times ... seven hundred and forty-one! (Sounds surprised)
68. R: Is that for (a) or (b)?
69. D: That's for (b)! I think (a)'s wrong actually, but anyway ... *(Assessment—accuracy of result)* (Long pause, writing. Goes on to Question 19c.) C ... one ... C four ... is thirteen times ... eight thousand two hundred, no, eighty-two thousand two hundred ... (Long pause, writing. Responds to inaudible question, from student off camera.) Well we don't have any answers, so we don't even know if we're right. *(Assessment—accuracy of result)* (Continues working) Thirteen C two ... (now doing Question 19c)

Although Dylan did not verbalise all his working, it is clear that he was pursuing the correct approach to solving parts (b) and (c) (see Moves 67 and 69, and Figure 3).

The Sequel

The episode parsing analysis concludes here, because the original group of students now broke up to work on different questions. However, later exchanges between Dylan and Rhys provide further evidence that Dylan had reasoned out the correct strategy for dealing with combinations problems like Question 19. Rhys had lagged behind the other three boys, and only belatedly sought to compare his own methods with theirs.

122. R: Dylan, what did you get for 19 (c)? (Reads Dylan's working) Thirteen C one times thirty-nine C four ... (Sean and Alex talking at the same time. Rhys seems to be disagreeing with Dylan). That's what I reckon ... Shouldn't it be fifty-one C four? Because it says "at least one heart"; it doesn't matter if you have [inaudible]—
123. D: But there's thirteen hearts to choose from—
124. R: Yeah, but then the rest of the hearts could be chosen in the next, thing, so only one of them's there [inaudible]. There's thirty-nine to choose from, plus another twelve, which is fifty-one.
125. D: [inaudible]
126. R: (looking at Dylan working) Woh, man, our answers are so way different now.
127. D: You're stung with my answers, aren't you!

For Question 19 (c), Rhys argued that the condition of "at least one heart" in a hand of five cards is satisfied by selecting four cards from the fifty-one cards remaining once the required heart is removed. Although Dylan did not yield to his friend (see Move 123), he appeared to be unsettled by the difference in their answers, and the lesson ended with all students still unsure whether they had found the correct way to approach these problems.

Metacognitive Function of the Dialogue

The numbers and types of metacognitive acts were recorded for each student, and for the dialogue as a whole. Of the 69 Moves in the portion of the transcript subjected to analysis, 22 were coded as having a metacognitive function. Dylan and Alex were the main contributors, sharing ten of the eleven New Ideas (six for Dylan, four for Alex) and all eleven Assessments (seven for Dylan, four for Alex). However, this quantitative representation does not tell the full story, since it obscures an important quality of the students' metacognitive activity while working on the combinations problems—their inability to make valid judgments about their strategies and answers. For example, three of the four Assessments of understanding revealed that the Alex and Dylan *did not understand* how to approach Question 19 (Moves 26, 27, 58). Similarly, Dylan was *unsure* of the appropriateness of the strategy he implemented for Question 19 (a) (Move 24), and he expressed his *doubts* as to the accuracy of his answers (Move 69).

Another finding which deserves comment is the absence of any Assessments of the *sense* of results. The sheer size of the answers to combinations problems often astonishes students, and seems to contradict their expectations of what a reasonable answer would be. Lack of prior experience with combinations calculations may have contributed to the boys' difficulties in deciding whether or not their answers were acceptable.

Individual Interviews
Individual interviews were conducted with all the students who appeared in the transcript (coincidentally, on the same day as the lesson from which the transcript was obtained). Relevant questions, the metacognitive "red flag" they targeted, and student responses are summarised in Figure 4. Clearly, the students had access to a wide range of metacognitive strategies which could have helped them to deal with the type of impasse they faced in solving the combinations problems (see Question 1). Nevertheless, their dependence on the quality of their answers in making judgments about solution strategies (see Questions 2 and 3) is significant in this context, where the answers they obtained were difficult to assess for accuracy and sense.

Interview Question	Response Category	Examples of Student Responses
1. What do you do when you get stuck on a problem? *Lack of progress*	Assess appropriateness of strategy	Keep going back to the beginning, go through your reasoning again. (Alex)
	Change the strategy	See if there's another way I can do it. (Dylan)
	Identify new information	Try real examples that I know are going to work so that I know what the answer will be. (Rhys)
	Seek help from teacher or peers	I'll talk to my friends. If they're stuck as well we go and see Mr G (teacher). (Dylan)
2. How can you tell you've solved a problem correctly? *Anomalous result*	Assess result for accuracy and sense	You can check it another way or work backwards. (Alex) Just whether it makes sense. (Dylan)
3. How do you decide whether to change your approach to a problem? *Lack of progress* *Error detection* *Anomalous result*	Assess result for accuracy and sense	You can look and see if the answer makes sense so far. (Alex) You just think it seems a bit strange, or it's too easy. (Dylan) If you're getting real weird numbers. (Rhys)

Figure 4. Responses to interview questions

Discussion

This paper has been concerned with metacognitive failure, and the circumstances under which peer collaboration—often beneficial in helping students to clarify, justify and evaluate ideas—fails to achieve a mathematically successful outcome. It was argued that students can be misled by spurious metacognitive "red flags", which falsely warn that something is amiss, or raise doubts about the validity of the solution method. Such was the case in the lesson discussed here, when students found it difficult to judge whether their strategies were appropriate or their answers made sense, even when the solutions produced were correct.

Complementary data from individual interviews and the transcript of collaborative problem solving showed that, collectively, the students possessed a useful metacognitive repertoire which they attempted to bring to bear on the problems they had been set. Their lack of success was measured, not in terms of failure to solve the problems, but by their inability to resolve the uncertainty which had brought them together in the first place. Dylan did correctly solve Questions 19 (b) and (c), but the status of his solution remained doubtful in his eyes because it could not be validated by appealing to the authorities usually available to him—the teacher, a text, and his peers. In these circumstances, collaboration was metacognitively fruitless because the students did not have access to the means of resolving their uncertainty (Clarke & Helme, 1997). To begin with, they lacked *prior experience* with strategies for counting combinations, and they had no *empirical basis* for evaluating the accuracy or sense of answers which seemed improbably large. Neither were *knowledgeable others*, in the form of the teacher or peers, available or able to provide help. Finally, potentially useful worked examples in the *text* were sought out but overlooked.

Long term observation of the students participating in this study has provided evidence of the metacognitive benefits of collaborative interaction. Nevertheless, peers are only one several sources of authority to which students may appeal when making judgments about the validity of their solution strategies and results. This paper has highlighted the need to examine peer interaction in the context in which it occurs, that is, the classroom community in all its complexity, if progress is to be made in understanding the implications of collaboration for students' mathematical thinking.

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