Investigating students' understanding of the relationships among quadrilaterals

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This study is the second in a series designed to explore how class inclusion concepts evolve. Class inclusion concepts in this context is the ability to have an overview of possible relationships that exist among figures, and is an important characteristic of Level 3 thinking in the van Hiele Theory. Indepth interactive interviews were undertaken with 24 capable secondary students concerning how they view linkages among six different quadrilaterals. A developmental pattern is identified.

This paper presents the results of the second phase of a larger study involving a focused analysis of the understanding and development of class inclusion notions, in a geometrical sense. This is carried out within the framework offered by the van Hiele Theory (van Hiele, 1986). This theory hypothesises a five-level framework to describe students' thinking in Geometry, with class inclusion considered an important component of Level 3 thinking. This is a level where students are able to achieve an overview of the properties of figures, and relate (or order) the properties so that particular properties flow from, or are a consequence of, an individual property or group of properties.

While it is recognised that the development of class inclusion concepts is a prerequisite, which strengthens and deepens students' understanding for deductive reasoning, studies have highlighted the difficulty of the task faced by students in achieving and appreciating sub-class relationships (Burger & Shaughnessy, 1986; Fuys et al, 1985; Mayberry, 1981; Pegg & Davey, 1991). Some studies, which have measured levels of thinking via a written test, assign Level 3 thinking to a response if the student accepts a single statement involving class inclusion with no means for justification (e.g., Usiskin, 1982). Other studies (Burger & Shaughnessy, 1986; Pegg & Davey, 1991), involving short answer questions and interview tasks, have enabled a more detailed identification of the components of Level 3, highlighting that importance should not be placed on "the acceptance of class inclusion but the willingness, ability and the perceived need to discuss the issue" (Pegg, 1992, p.24).

To date, some limited work has been undertaken in the general area of class inclusion associated with quadrilaterals. However, it has basically revolved around answers to such questions as those which require the recognition of squares as rectangles (Usiskin, 1982; Mayberry, 1983). In contrast, an earlier stage of this current research, as presented at MERGA 20 (Currie & Pegg, 1997) addressed an investigation of students' understandings of class inclusion, as related to different triangle types. The results of this study revealed that class inclusion appeared to be a gradually acquired skill which follows a developmental path. The hierarchical framework began with links based on the spontaneous identification of a single similar feature or property, without the formation of triangle classes. Following this, three classes of triangles emerged and became a workable identity. Further growth was indicated through a stronger focus from each class as separate identities, to the relationships that exist between classes. The culmination being the ability to incorporate subsets.

These results act as a useful preliminary baseline for the current study due to the less complex structure of relationships between triangles as compared to quadrilaterals. Also, the design features used in the previous study which included indepth task orientated interviews were seen as very appropriate for clarifying students' attempts at trying to understand the relationships between six different quadrilaterals.

Three research questions guided the study. They are:

1. What was the nature of the links students formed when grouping different quadrilaterals?
2. Was there evidence of some developmental pattern in the structure of the different responses?
3. What similarities exist between the nature of the links students form among triangles and those formed among quadrilaterals?

Design

Twenty-four students, six from each of Years 8 to 11, were selected from two secondary schools in Armidale. The students were of above average ability, and there were equal numbers of males and females.

The purpose of this study was to have the students identify and justify relationships among six quadrilaterals, namely, square, rectangle, rhombus, parallelogram, kite, and trapezium. The interview format, as contained in Table 1, outlines the student tasks and questioning focus which enabled the students to work with familiar recalled information, supplemented information, individual tree designs, and discussion involving prompts and probes from the interviewer. The continual revisiting of the same issues, in slightly different forms, as drawn on different maps by the students, provided a vehicle for extracting further information, as the maps were used as a catalyst for discussion concerning the reasons for the existence of links (or relationships). The investigation required the analysis of responses to questions (ii), (iv), and (v) in Table 1 which deal with the relationships between the six quadrilaterals.

Quadrilateral Relationships	
(i) Int:	Write a list of all the quadrilateral names you can think of. Draw each quadrilateral.
(ii) Int:	Design a tree diagram which links the different quadrilaterals. Draw a sketch to link each type. (Discussion to follow concerning the reasons for links or lack of links. If required the following three points are addressed.)
(iii) Int:	There are some quadrilaterals that we can add to this list (names of quadrilaterals not recalled are provided). Draw a sketch of each new quadrilateral.
(iv) Int:	Design a second tree diagram incorporating all quadrilaterals on the list. (Discussion follows concerning the reasons for links or lack of links.)
(v) Int:	Taking your first tree add the new quadrilaterals to your original tree, again include relevant links. (Discussion follows concerning the reasons for links and lack of links.)

Table 1. Summary Interview Structure

Results

Overall, the students found the ideas familiar, but the questions were seen as non-routine. Analysis of the twenty-four interviews, which considered the links described, and the justification of these links, made evident the significance of a developmental path and allowed a comparison between the growth identified previously with triangle relationships and that identified here with quadrilaterals. Below is a detailed analysis of the main categories of responses, including a description of one typical student response for each type, and relevant samples taken from the interview transcript.

Type A

These responses include links based on the identification of a single similar feature or property, such as 'at least one set of parallel lines', 'equal sides', 'right angles', 'two sets of equal length sides', and links based on some aspect of a given shapes appearance. While the groupings of quadrilaterals formed in this way are based on a unifying characteristic, the most telling feature is that a particular quadrilateral may occur in several different groups. There were three responses coded as Type A. The differences evident in the students' responses were attributable to a gradual decrease in links based on overall appearance and an increase in links associated to different properties.

One of the better Type A responses was provided by Tracy. She was able to form groups on the spontaneous identification of 'parallel sides', 'right angles', 'diagonal parallel lines', and 'two sets of equal lines'. Tracy's discussion below illustrates her concentration on a single particular feature or property to link the figures. When prompted to discuss other links, Tracy accepts that they may connect for more than one reason, but forms the new group in isolation to links identified previously.

Int:	How does your kite link to the square?
Tracy:	Because they both have two sets of equal lines which means that those two are the same. This is going to get really messy. I have got all those ones that have at least one set of parallel lines.
Int:	Can you tell me about these other ones?
Tracy:	Well this is just connecting the ones that have two sets of equal length sides, that is, already linked.
Int	Which one do you mean?
Tracy:	Well that is linked because it has parallel diagonal lines.

Overall, a Type A response is characterised by the spontaneous formation of groups based on the identification of a single unifying feature or property. The properties of the quadrilaterals are treated in isolation, and links based on other properties are not considered within that group of quadrilaterals. The groupings change as frequently as the identifying feature changes.

Type B

These responses appear similar to Type A in that links are based on similar properties. However, the lack, or existence, of a perceived significant property precludes a given link. For example, whereas in Type 1, squares and parallelograms would be linked because of parallel sides, in this grouping they would not be linked because of the absence of a right angle in the student's image of a parallelogram. This comes about because in this grouping, the particular named figure, such as a square, stands for certain known properties. For example, links, such as square to rectangle, square to rhombus, rectangle to parallelogram, parallelogram to rhombus, parallelogram to trapezium, will be made on the basis of similar properties, such as right angles, parallel sides, equal sides, and opposite angles equal. When describing the links identified, the student also articulates the differences between the shapes which preclude certain links. An important feature of the eight responses coded as Type B, is that a link is not made between the parallelogram and the square, or the rhombus and the rectangle, unless students are prompted to do so.

Andrew's response illustrates the links typical of a Type B response. The names of known shapes is represented by properties. The differences between the shapes are expressed as well as the similarities, and negative instances preclude the formation of links.

Andrew describes the properties in restricted terms, namely: 'the sides are longer'; 'length and angles are the same'; and, 'almost the same length and parallel'. The properties of the parallelogram and rectangle are described using language which restricts the inclusion of links based on similar properties. For example, concentration on right angles, and the 'same length' is utilised, and does not include opposite angles equal and opposite sides equal.

Int:	Do you see the square connecting to the rhombus at all?
Andrew:	Yes because they are all the same. See they are the same length and they are parallel and they have the same angles. Except that one is 90° and that is about it ...
Int:	Tell me about your rhombus to your parallelogram.

Andrew:	Well they (rhombus and parallelogram) are almost exactly the same except one (parallelogram) has only one side as longer, and that (rectangle) can go to that (square) because that is square but one side is longer so that is just like that.

In summary, Type B responses link quadrilaterals on the basis of similar properties while acknowledging a number of the properties represented by that shape. Although links may exist due to one similar property, other properties of the quadrilateral are also noted, and either reinforce the connections, play no role in the connection, or are dominated by negative instances over other similar properties. For example, the rectangle would not be able to link to the rhombus, as the rectangle is described as having two sets of equal sides, where the pairs are of different lengths, and, the rhombus has all sides equal. Another dominating difference, which precludes this link, concerns the rectangle being described as having four right angles, while the rhombus is described as having two obtuse and two acute angles.

The responses in this category are also characterised by restrictions placed on property descriptions. For example, an inability to describe the square as having opposite angles equal, or opposite sides equal precludes its link to the parallelogram. The majority of responses in this category also include responses based on overall appearance which accentuate the differences between shapes.

Type C

Type C responses are very similar to Type B descriptions with the addition of tentative links suggested between the square and the parallelogram. Similar properties are noted and described in a more inclusive manner. However, the differences expressed result in an indecisiveness concerning this relationship. There was only one student (Alice) coded as Type C. This response illustrates the conflict evident between inclusive descriptions of figures, and the differences identified between them.

The significant difference between Alice's response and the Type B responses lies in Alice's ability to discuss the possibility that the parallelogram can have four right angles, as the sides will still be parallel. Alice continues by stating that if the sides were the same, the parallelogram would be a square. When prompted to discuss the possibility that the square is a parallelogram, Alice is restricted by other properties of the square and describes parallelism as an unimportant characteristic of the square.

Int:	So do you think that they can go together? (parallelogram and square)
Alice:	Um, no it is not important enough. That one (rectangle) there has got four right angles so it can go to the square. That one (parallelogram) can have four right angles as well.
Int:	The parallelogram?
Alice:	Yes because all of these sides are parallel and these sides are parallel and if they are all the same then it is a square.
Int:	So does that mean that the square is a parallelogram?
Alice:	No because parallel isn't important to a square.

The indecisive nature of a link between the square and the parallelogram coincides with a willingness to describe the parallelogram as encompassing right angles and four equal sides. The square, however, is restricted in nature and is dominated by the right angle property, hence, the link remains only a possibility.

In summary, the Type C response includes links based on similar properties while taking into consideration known properties of each figure. The differences between the figures are not as dominant in nature compared to the Type B responses, allowing a tentative link to be made between figures, such as the parallelogram and the square, which in Type B would have been precluded. The parallelogram is described in less restricted terms, thus, acknowledging the possibility of including right angles and equal sides.

Type D

This group of responses is characterised by the addition of links which were not included in the Type B responses, and which were tentatively linked in the Type C response. Differences observed, such as right angles and equal sides, do not hinder the links that exist between the parallelogram and square, and/or, the rhombus and rectangle. These links are described in terms of inclusive language. For example, students use, 'opposite sides parallel', 'opposite angles equal', and 'two sets of equal sides'. Links exist between all parallelograms in the majority of responses based on similar properties, but the classes of parallelograms, rectangles, and rhombuses, are not inclusive of other classes of figures. For example, the rhombus, rectangle and square are linked to the parallelogram but none are seen as a subset of the class of parallelograms. There are four responses coded as Type D. A typical Type D response is provided by Jenny's answers.

Jenny's response illustrates the use of language which allows the formation of more links when compared to the Type B and C responses. Jenny's response is not focused upon differences between the quadrilaterals, thus, similarities described in terms of properties are not dominated by the properties with strong visual cues. Jenny acknowledges that the square has opposite sides equal and parallel, but does not describe the link in terms of opposite angles equal.

Int: Can you tell me why you have the square, the rhombus, the rectangle and the parallelogram all linking up together?

Jenny: Well it is more of a progression I suppose as the square has all right angles and so does the rectangle that is why they are linked. And also they have opposite sides equal and parallel lines, the opposite sides are parallel. With the rhombus all the sides are equal and the same with the square and the parallel lines also um these are obviously not right angled as this doesn't have right angles either but the opposite sides are parallel and the opposite sides are equal. These two are linked because everything is the same except this one doesn't have right angles.

When prompted to include the square, rectangle, and rhombus within the parallelogram class of quadrilaterals, Jenny accepts that they are a group but is unable to place these shapes within the generic category carried by the word parallelogram.

Int: Do you see these as fitting into a group together?

Jenny: Yes.

Int: What would you call it?

Jenny: I don't know.

In summary, Type D responses are characterised by the addition of links between the parallelogram and square, and/or the rhombus and rectangle. The links based on similar properties acknowledge all the known properties of the generic category represented by a quadrilateral name. The links are not hindered by the properties dominated by appearance, and the descriptions encompass a range of property subsets as opposed to the exclusion evident in the Type B responses. Each of the responses in this category give none or few justifications based on overall appearance. This emphasises the reliance on the known properties of the different quadrilateral types, and the reconciliation between similarities and differences made possible via the inclusive nature of the language used.

Type E

This group of responses includes the class of parallelograms as an important feature of the quadrilateral relationships. The class of parallelograms is described succinctly, and is based on similar properties. The parallelogram encompasses other classes of figures such as square, rectangle, and rhombus, with justifications based on more than one similar property. There are links made to the trapezium and kite which are based on similar properties. There are three responses coded as Type E. The links and justifications described by David are typical of a Type E response.

David's initial comment concerning the design of the tree diagram indicates that the class of parallelograms is comprised of subsets. David is not content with his diagram, as it suggested that the square and rectangle are not part of the parallelogram class. He then alters his diagram to rectify this problem.

> David: I was going to start with has right angles and doesn't have right angles but a parallelogram would end up over there, but a rectangle and a square are parallelograms, but I don't think I can differentiate between the two because they are parallelograms because they do have parallel sides. This is a tricky one ...

David's discussion demonstrates an ability to distinguish between the different quadrilateral types within the class of parallelograms. He describes the properties characteristic of the class of parallelograms using inclusive language, such as, 'opposite sides parallel and equal', and 'opposite angles equal'. David is able to maintain a distinction between figures represented by different names through the use of language such as 'all sides equal'. When prompted to address the class of parallelograms, David describes in detail the justification for this, and states that the square is a parallelogram but the converse of this is not true.

> Int: Can you tell me why this parallelogram has all these lines coming off everywhere?
>
> David: Because it has almost all the characteristics of the other shapes like it has the parallel sides, the set of equal opposite sides, it can have angles that equal 90°, but it can't be um, you have to remember that a square is a unique parallelogram, as in a parallelogram can't have sides of equal length. As in a parallelogram can be a rectangle but it can't be a square.
>
> Int: Can a square be a parallelogram?
>
> David: Yes it is a parallelogram.

David's response accentuates the links between the rectangle and square, and rhombus and square, by describing the additional similar properties, however, the notions of the rhombus class inclusive of the square, and the rectangle class inclusive of the square is not evident.

In summary, Type E responses are characterised by the formation of a parallelogram class inclusive of other quadrilateral types categorised by a different name. The language used to describe the relationships is inclusive, while different figures within the class of parallelograms are distinguished by their additional properties.

Type F

This type of response includes the class of parallelograms characteristic of a Type E response, with the addition of the rectangle class, inclusive of the square, and/or the rhombus class, inclusive of the square. The justifications for these are based on similar properties with links also made to the trapezium and kite. There is only one response coded as Type F. Brendan's response includes: the class of parallelograms which encompasses the square, rectangle, and parallelogram; the class of rectangles which encompasses the square, and, links based on similar properties.

Unlike David, Brendan begins his discussion of the tree diagram by describing in detail the individual links between the different quadrilaterals using similar language to the Type E responses. When Brendan is prompted to discuss the parallelogram class, as suggested in the design, Brendan begins with a description of the rectangle class as incorporating the square. The rhombus is described incorrectly as a special square which suggests that the rhombus subset is evolving. The parallelogram class is described as the basis from which the square, rectangle, and rhombus come from.

Int: Is there a reason why these three shapes all go to the parallelogram or why these three all go to the rectangle?
Brendan: It is do with the parallel sides and a square is a rectangle but it is special and the same as a rhombus is a square but it is a special one.
Int: What about your parallelogram?
Brendan: A parallelogram, um, a rhombus certainly comes from a parallelogram and as I say a square is also a rhombus.
Int: Is a square a parallelogram?
Brendan: Yes.
Int: What about a rectangle?
Brendan: Yes it is but it isn't flat ...

Although Brendan's description of the classes with subsets are not described with the same detail as David, the nature of the links formed, when compared to a Type E response, are more complex. The class of parallelograms includes other figures while also mentioning subsets of the rectangle class, however, Brendan describes incorrectly the rhombus as a subset of the square class.

In summary, the Type F response is categorised by the class of parallelograms including the subsets of square, rectangle, and rhombus. Within the class of parallelograms, two classes exist which incorporates subsets, namely, the rhombus with the square subset and/or the rectangle with the square subset. Each of these classes can be justified on the basis of similar properties.

Summary

The students' responses were coded into six types. These can be summarised as;

Type A: One single similar property identified to link the quadrilaterals. The groupings depend upon the unifying feature chosen and this property remains the single focus of the link.

Type B: The formation of quadrilateral classes known by name and categorised by a group of properties. The classes have no subsets. Relationships are described between classes when a perceived significant property is evident which often has visual cues. Links are described using language which often precludes links. For example, four equal sides does not encompass opposite sides equal. Links do not exist between the square and parallelogram, and rhombus and rectangle.

Type C: Characteristic of a Type B response with a tentative link made between the square and parallelogram. The similarities are noted between the shapes but the differences observed hinder the formation of a link.

Type D: The relationships are formed across the quadrilateral classes and are based on similar properties using inclusive language. The link is made between the square and parallelogram, and, the rhombus and rectangle, as the properties have developed an encompassing quality. Relationships are not dependant upon visual cues.

Type E: The class of parallelograms includes the figures known as square, rectangle, and rhombus. These subsets are justified on the basis of similar properties.

Type F: The class of parallelograms described in Type F acquires further development. The subsets also contain figures known by a different name. Within the parallelogram class the rectangle is inclusive of the square, and the rhombus is inclusive of the square.

When this overview of the results of the different types of responses are compared with the results of the previous study, involving relationships among different triangle classes, three issues stand out. First, there was a marked similarity between the findings of the two studies. In both cases, clear categories were identified as were transitional responses where students appeared to be struggling for a more sophisticated answer. Associated with this, is that the similarity occurred even though there was a subtle difference in focus. In the case of the study involving triangles, a number of the triangles were variations of a specific class, e.g., there were three types of scalene triangles, namely, acute-angled, obtuse-angled and right-angled triangles, whereas in this current study the focus was on general representatives of a class. Second, early responses in both studies saw the transition from a shape having certain characteristics/properties to the properties characterising the shape. Third, certain descriptions based on known properties precluded the formation of links in the case of different triangle classes (e.g., three sides equal did not prompt two sides being equal), this was also evident in attempts to link quadrilaterals (e.g., four sides equal did not prompt opposite sides equal).

Conclusion

Overall, the results of this study provide substantial support for the tentative findings of a previous study of a developmental pattern in the acquisition of class inclusion concepts. Also highlighted are the difficulties that students must face and the cognitive hurdles that must be overcome to reach such an understanding. The good news is that with this information, teachers may have, for the first time, a structured framework in which to plan ways forward to help students develop such skills.

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