THE BIG IDEAS IN TWO LARGE FIRST LEVEL COURSES OF UNDERGRADUATE MATHEMATICS

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What is important to teach students within the mathematics discipline? Identifying the fundamental concepts (or big ideas) of mathematics is being looked at in the development of the Australian National Mathematics curriculum. But what do lecturers at university consider to be the "big ideas" in the mathematics courses they teach? Seventeen lecturers were interviewed about thirteen mathematics courses to establish what they considered were the key areas of learning within these courses. This paper reports on the interviews conducted with lecturers from two large first year classes. Their responses indicate that teaching mathematics involves a lot more than mathematics alone.

Introduction

Currently Australian schools are preparing to adopt a National Curriculum for four subjects, including mathematics. The *Australian Curriculum—Mathematics* is intended to focus on the fundamental concepts in mathematics that should be taught. Establishing what the "big ideas" or mathematical concepts that students need to grasp in mathematics is also relevant in a university setting.

How well a student grasps a new concept can be described in terms of *Concept usage* which matters in mathematics, as students are not only required to understand mathematics concepts, but also to use them in processes that require other abilities such as the use of logic and critical understanding (Moore, 1994). *Concept image* is the term used to describe how a concept is understood and seen by the student (Tall & Vinner, 1981). The *concept image* is developed over many years and is influenced by the student's experience from within and beyond their education. The *concept image* develops either consciously or subconsciously with more experience. Thus the student may have a very different understanding of the concept from that held by their discipline. The discipline's understanding of the concept is described as the *concept definition*. For example, when it comes to developing formal mathematical proofs, understanding of *concept usage* becomes important (Moore, 1994). Without this understanding, the ability to use proof techniques diminishes. For any form of mathematics, *concept definition*, *concept image* and *concept usage* are all important factors.

Some concepts have a greater impact on a student's learning and can be described as threshold concepts. A threshold concept is a term that has emerged within the literature on higher education learning and teaching as a way of thinking about how students come to understand the key ideas of their discipline. The theory of threshold concepts was initially developed by Meyer and Land during a national research project in the UK in the economics discipline (Cousin, 2006). It was believed that understanding some crucial concepts, which they called threshold concepts, was essential to becoming an economist (Cousin, 2006). The theory of threshold concepts has now been taken up and developed by many other disciplines including mathematics.

Each mathematical concept can be described by the *concept definition,* applied according to the *concept usage* and seen by the student through their *concept image.* However, these concepts can be categorised into two levels of importance: core concepts that define important stages in learning and threshold concepts that, once grasped, will change the way the student thinks (Meyer & Land, 2003). Threshold concepts are most likely to be transformative, irreversible, integrative, bounded and troublesome (Meyer & Land, 2006).

Once a threshold concept is grasped, the student is said to gain new insight into what they are studying so that the material they are working on becomes clear and obvious. They are therefore changing from the *concept image* they possess to an understanding of the *concept definition* held by the discipline. In mathematics, complex numbers, limits, proofs and calculus have all been described as examples of threshold concepts (Easdown, 2007; Meyer & Land, 2003; Pettersson & Scheja, 2008).

The purpose of this study was to identify the "big ideas" that mathematics lecturers want their students to learn. For ease of communication, the term "areas of learning" was adopted in interviews to describe these "big ideas". These areas of learning will be presented and discussed in terms of the core versus threshold classification, as well as in terms of the students' progression through concept image to concept definition and concept usage. The areas of learning also include skills that cannot be classified as mathematical concepts.

Research design

The interviews discussed here are part of a larger integrated study being conducted at The University of Queensland that will use interviews, student surveys and analysis of course assessments to investigate how students' behaviour and attitude affect their ability to understand key concepts in mathematics.

The lecturers were sent the following list of questions before each interview:

1. Please list four to five main areas of learning in the course. These can be concepts, skills, or topics that you consider of key importance for the students to attain by the end of the course.
2. Are any of these key concepts similar to those in other courses?
3. Do you feel that students are made aware of the key concepts they are expected to understand?
4. What do you do when teaching the course to aid the students in gaining understanding of these concepts?
5. Does the assessment in the course encourage students to gain understanding of these key concepts?
6. Does the course assessment determine whether a student has understood these key concepts?

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6. Does the course assessment determine whether a student has understood these key concepts?

7. How do you think the students' grades reflect their understanding of the course material (and whether they have understood the key concepts)?

Initially, responses were recorded by taking hand-written notes that were sent back to the lecturer for verification. Later interviews were audio recorded, which allowed for greater accuracy and depth of information collected.

The majority of lecturers interviewed have had extensive teaching experience in a variety of universities and had taught the courses for several semesters.

Results

Interviews were conducted with 17 lecturers. Two courses were selected for analysis in this paper. The lecturer who taught the first course, *Mathematical Foundations*, identified areas of learning similar to those in other courses, but gave very different reasons for his choices. The second course, *Multivariate Calculus & Ordinary Differential Equations*, is typical of courses taught by two lecturers.

Mathematical foundations

Mathematical Foundations is a course that caters mainly for engineering students who have not completed advanced mathematics at secondary school. The pace of mathematics covered is quite fast for some students as the content usually covered in two years at school is taught in one semester (13 weeks) at university. In the first semester of each year enrolments are around 700 students.

The lecturer, Tom, identified the following four major areas of learning:

1. Limits;
2. Recognition of different number systems;
3. Proof by induction; and
4. Describing physical problems in the language of mathematics.

He pointed out that his intention was not to teach students detailed procedures in each area, unlike many other interviewees, but to introduce them to mathematical thinking; for example, he wanted students to understand how and why different number systems were introduced. This was to show them the relevance of the different types of numbers in the course, such as complex numbers.

Tom wanted students to be able to describe physical problems in the language of mathematics. He explained that this skill showed the usefulness of mathematics in solving "real life problems". It also brings together the areas of mathematics in the course. Showing students the relevance of what they are learning has been shown to encourage students to adopt a deep learning approach (Entwistle & Tait, 1990). These four areas of learning are all used extensively in later courses, where they are classed as assumed knowledge.

Tom felt that all the areas of learning were tested for understanding in the course assessment. He generally found that, although most students understood limits, their algebra skills were poor. In the final exam, which was regarded by Tom as quite difficult, number systems (in the form of complex numbers), proof by induction and applications were always tested. Tom said that a student would need to know well all the areas of learning to achieve a high distinction. However some students could pass the course by only understanding one of the areas of learning. A credit could be achieved if that area were applications. It is not possible to gather which key areas of learning a student had understood only by looking at their final grade.

Tom's descriptions show that the course was not just about learning concepts but more about using the concepts to teach the students how to think and act like mathematicians.

Multivariate calculus and ordinary differential equations

This is a large first level course that usually has around 1000 students enrolled in the second semester. As with the previous course, the majority are engineering students, but there are also some students majoring in mathematics. The course complements previous courses by introducing students to more advanced aspects of calculus. They solve a variety of problems involving functions of several variables, partial derivatives and parameterisation of curves and line intervals.

Due to the large number of students enrolled, there are often several lecturers teaching the course. The two lecturers interviewed, Bob and Alice, have each been teaching this course for several semesters. Bob's responses relate only to the part of the course that he taught, whereas Alice's responses relate to the entire course. Alice teaches the later part of the course with much of the material related to the early part.

Table 1. Areas of learning: the analysis of Bob's and Alice's responses.

Analysis: Areas of learning	**Bob's responses**	**Alice's responses**
Critical thinking	Critical thinking	
Applications	Applications: to motivate and show how calculus can be applied	Using ordinary differential equations in modelling applications.
Graphical interpretations of plane interception	Linear algebra: looking at how planes intercept	
Vector calculus		Basic understanding of line integrals and the use of parametric curves in their evaluation; parametric representation of curves in 2 and 3 dimensions.
Computational aids		The use of *Matlab* or other computational aids to assist in visualisation/understanding of concepts.
Understanding functions of more than one variable	Differentiation of functions of more than one variable Graphing functions of more than one variable by visualising graphs from functions, using geometric interpretation to make predictions	Linear and quadratic approximations to functions of more than one variable Using partial derivatives to analyse key features of functions of more than one variable. (e.g. tangent planes, max/min problems, Lagrange multipliers) Rates of change of functions of more than one variable, interpreting this in graphical terms
Ordinary differential equations		Solving and interpreting solutions of certain ordinary differential equations

Critical thinking, applications, and computational aids are the three areas of learning that relate to the entire course. Bob stated that one of the most important skills he wanted to teach students was to critically think. As he stated, “No one should graduate from a respectable university without being able to critically think”. Though critical thinking is not exclusive to mathematics, Bob felt that many students in his course lacked this skill, which in turn affected their ability to engage in learning mathematics. Although Bob saw critical thinking as a major issue that needed to be addressed with his students, it was not mentioned by Alice.

Alice often uses *Matlab* herself to demonstrate problems in lectures. She would therefore see and demonstrate the value of *Matlab* to her students. Bob did not see the necessary benefit of Matlab in the course. During the interview he mentioned that, though Matlab was meant to be an additional tool to aid understanding, he doubted that this was achieved.

Both lecturers valued applications but saw their use very differently. Bob saw applications as a means to motivate students, whereas Alice saw applications as learning tools that will teach students more about the course content.

The other responses from both lecturers relate to the course content. Both placed emphasis on understanding and interpreting mathematics. Alice seemed to place great importance on understanding course content, whereas Bob appeared more interested in developing students’ mathematical thinking. The students taking the course would find a very different emphasis placed on learning from one half of the course to the other.

All these areas of learning are developed and used in more advanced courses. Both lecturers emphasise the areas of learning when lecturing; however, Alice also mentioned that all hers are in the course profile.

To aid students to grasp the areas of learning, Bob encourages student involvement and discourages students from sitting in the back two rows. He said his students are well behaved and happy to ask lots of questions. To emphasise critical thinking, he often tells his students that they should “Think, think, and think again”.

The weekly assessment for the course contained challenging questions which Bob hoped extended students and encouraged understanding. However, he was not convinced that this was achieved as many students struggled with these questions. Alice was more confident than Bob that students did achieve understanding.

The final examination was considered by both lecturers to determine students’ understanding of the areas of learning they had identified. Bob explained that the final examination had some challenging questions different from those seen in lectures or assignments. He expected that a high distinction would indicate that a student had grasped all the areas of learning, but conceded it was possible that students could pass without the ability to critically reason and with only understanding about half of the areas of learning. He considered the students who failed to be the ones who struggled with most of the areas of learning and made basic mathematical errors. Although both lecturers were confident that student grades correlated with the level of knowledge acquired, they were unsure whether their students had grasped any particular important area of learning they had identified.

Discussion

Analysis of the full set of interviews indicated that lecturers not only have very different ideas of what is important for students to learn within their courses, but also different ways of justifying common choices. So two lecturers may identify areas of learning as important, but for different reasons. For example, proof by induction was identified as a key mathematical concept in two courses, the mathematical foundations course described above and the first level discrete mathematics course. The same area of learning has very different purposes in each course. In the first course, it was to show students that mathematics is established on well-founded reasoning, and in the discrete course it was so that students would understand that proof by induction is an alternate method of proof that is not necessarily intuitive, but extremely useful. Sometimes the importance of the concept is what the concept demonstrates rather than the *concept definition* or *concept usage*. The way lecturers teach mathematical induction would create very different *concept image*s by the students.

The courses included in this paper also place different emphases on core versus threshold concepts. For example, it was not surprising to find that in the mathematical foundations course, three of the areas of learning Tom described are considered threshold concepts (Easdown, 2007; Meyer & Land, 2003; Pettersson & Scheja, 2008). Many of areas of learning from both courses can be seen as core concepts, threshold concepts or possibly as processes with more importance placed on their *concept usage*. This interpretation will depend strongly on how they are assessed. For example, within the topic of understanding functions of more than one variable, a student can be assessed on the process, or on their understanding, or on the definition. Some areas of learning involve abstract ideals not necessarily related to a particular concept. These are more like graduate attributes such as critical reasoning or being able to convert real life problems into mathematics.

Even within courses taught by more than one lecturer there are differences in emphases. In the multivariate calculus course, Bob's goal is to raise students' mathematical thinking ability and not just for them to grasp the course content. He felt strongly that students' lack of critical thinking skills affected their engagement in mathematics. However, taking mathematics courses has not been shown to increase students' critical thinking skills (Terenzini, Springer, Pascarella, & Nora, 1995), even though critical thinking skills can be increased by student involvement in other courses and outside their formative study (Terenzini, et al., 1995). Alice has a different goal, since she seems to place more emphasis on students mastering essential content and skills. These differences raise interesting questions about the relative emphasis in the two halves of the course on developing *concept usage*, *concept images*, and *concept definitions*.

These lecturers and many others interviewed indicated that the assessment processes they used did not allow them to know with confidence whether students – apart from those achieving high distinctions – had grasped the areas of learning they considered most important. This finding suggests that it is not enough to identify the "big ideas" for inclusion in a mathematics course; assessment tasks must be capable of eliciting these ideas from students in a way that lecturers can recognise.

This study is currently being expanded to identify the "big ideas" of second and third level courses. This will establish a more detailed picture of how mathematics is taught, developed and connected for the undergraduate student majoring in mathematics, which in turn can inform teaching.

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