DEVELOPING ALGEBRAIC THINKING: USING A PROBLEM SOLVING APPROACH IN A PRIMARY SCHOOL CONTEXT

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This paper reports on an ongoing research project investigating how problem solving can prepare students to think algebraically. The student examples presented highlight how investigating and solving mathematical problems from a structural and generalised perspective can develop the thinking associated with algebraic reasoning.

Introduction

For more than 50 years there has been a call by many experts in mathematics education research, curriculum design and policy development that students in primary school should learn and understand a level of mathematics beyond computational procedures. Directly related to this request has been a response to include algebra within the primary school curriculum. One reason why many western democracies have undertaken the challenge of reforming the primary school curriculum has been the steady decline in the participation rates of students undertaking advanced mathematics courses at a secondary school level (MacGregor, 2004; Stacey & Chick, 2004). Consequently, the declining participation rates and limited engagement with mathematics has slowly impinged on the availability of competent individuals who wish to, or are able to, pursue careers in the mathematical rich vocations offered at a tertiary level (Brown, 2009 p. 5). The inclusion of algebra in primary and middle school mathematics curricula reflects the belief that not only is algebra needed to participate in the modern world; it also provides "an academic passport for passage into virtually every avenue of the job market and every street of schooling" (Schoenfeld, 1995).

Currently, most primary and middle years mathematics curricula do not solely emphasise the teaching and learning of formal algebra. Instead, the emphasis in these formative years is about developing a conceptual understanding of algebra and in particular the thinking associated with "doing" algebra, often referred to as algebraic thinking. Algebraic thinking is the activity of doing, thinking and talking about mathematics from a generalised and relational perspective (Kaput, 2008; Mason 1996). Ultimately, algebraic thinking is founded on the ideas and concepts of elementary mathematics and in turn these ideas are used to solve increasingly sophisticated problems. It encompasses all mathematics strands and is built on a conceptual understanding of number and computational fluency, the reasoning of geometry and the processes associated with measurement and statistics.

The potential value for using problem solving is that it may broaden and develop students' mathematical thinking beyond the routine acquisition of isolated techniques and procedures (Booker 2007; Booker & Windsor 2010; Carraher & Schliemann, 2007; Kaput, 2008; Lins, Rojano, Bell & Sutherland, 2001). The thinking required to solve problems can be extended from methods tied to concrete situations—the backbone of primary school mathematics—to experiences that develop an ability to problem solve using abstractions. To consider problems from an algebraic thinking perspective acknowledges that students can adapt their ways of thinking, they can express mathematical generalisations and it can provide an entry to algebraic symbolism that is meaningful (Carraher & Schliemann, 2007).

Research project

The aim of the research project reported here is to explore and gain insights into the effectiveness of using a problem solving approach that facilitates and promotes certain aspects of algebraic thinking. The research aims to provide an improved and deeper theoretical understanding of algebraic thinking and how it can be developed within a primary school context. The intention of the investigation is to develop and implement lessons that actively facilitate algebraic thinking by building on students' problem solving experiences. Furthermore, the research project will seek answers to the following questions:

1. Can problem solving be used to develop algebraic thinking in the primary school context?
2. To what extent are primary school students equipped to use algebraic thinking strategies when solving mathematical problems?
3. What is the effect on students' ability to move from arithmetic to algebra, once a broad problem solving approach that explicitly develops algebraic thinking has been implemented?

Methodology

Part of this study is set in a Year 7 class in a State Primary School that draws from a pre-dominantly lower socio-economic background. Within the cohort of 27 students there is a wide variation in their understanding of mathematics and this position is supported by their 2010 National Assessment Program – Mathematics results. Furthermore, results from Booker Screening Tests (Booker, 2011) re-confirm the diversity and wide ranging mathematical abilities within the group. It would be reasonable to suggest that this class reflects many of the difficulties, challenges and rewards those classes and schools in similar socio-economic areas deal with on a daily basis.

This qualitative research project uses the method of design research and is greatly informed by the research methodology developed and used by Cobb (Cobb & Bauersfeld, 1995; Cobb, 2007). A key aspect of Cobb's interpretation of design research is the importance of collecting primary sources of data by observing and registering mathematical activity by the participant observer/researcher. In addition to this, Cobb also argues that by constantly reflecting on participant actions and synthesising the data a cycle of enactment, analysis and further refinement can allow for generalisations about learning based on all the different elements found within classrooms. Within the context of this study, student work samples and digital video recordings of individual students, small group interactions and whole class presentations and discussion, form the basis of the observations to be analysed.

Findings

This is an ongoing study and it must be noted that the analysis of the data is in its infancy however, the view that has emerged in the early analysis and based on the types of problems the students can solve, the approaches they have adopted and the way they have discussed and presented their results, indicate a growing ability to consider mathematical problems from an algebraic thinking perspective.

At the beginning of the study, it was hypothesised that students would need to develop ways of thinking that moved them from the computational thinking that dominates much of their enacted primary curriculum. A major hurdle to overcome within the cohort was an assumption and behaviour that to solve mathematical problems simply requires numbers to be manipulated. To reduce the influence of this perception, one of the foci of the study was for students to share with their classmates the reasons why and how they developed their solutions. The emphasis to share their mathematical reasoning was a powerful way to motivate the students. By encouraging them to develop a variety of different solutions they began to see the interconnectedness of the mathematics, which in turn influenced their ability to generalise their solutions. The discourse and argumentation that took place assisted individual students to reflect on, modify and delve into all of their mathematical knowledge in order to solve the problems. The opportunities to discuss and exchange mathematical ideas allowed many of the students to overcome the behaviour of calculating using the numbers from within a problem. One particular student's explanation for solving an assortment of structurally related problems was indicative of the way many of the students began to think about the problems. No longer did students immediately try a guess and check method but they attempted to find a generalised approach to the related problems.

> *Nikki:* It's something you can just do for everything ... I've done the problems before but I have never really thought about them. I can do all these problems now because I know a way that works for all of them.

Setting the stage – An overview of the lessons

There is a degree of consistency with regard to the implementation of the lessons throughout the research project, with each 45–60 minute lesson following a similar cycle. Each lesson was introduced with a whole class question where each group, usually made up of 4 students of varying mathematical abilities, were given the same question. After each group had completed the question they had to explain their solution to the researchers, classroom teacher or peers. In preparing their explanation the group had to consider "Why do you think you are right?" which directed them to address their thinking and mathematical ideas, rather than "How did you do it?" which emphasises the procedural steps to solve the problems. The next part of the lesson cycle involved giving each group a contextually different yet structurally similar problem. With each new question the mathematics became increasingly more complex Depending on the difficulty of the problems most groups would complete between two to four problems per lesson. The lesson would conclude with a whole class discussion in which students would present their solution.

It cannot be over-emphasised how important the group and class discussions were in igniting and developing different mathematical ideas. During each part of the lesson cycle the collaborative manner in which many of the students conducted themselves was highly productive. Clearly, they engaged with the mathematical discourse of their peers and this had a profound effect on their mathematical thinking. Within the context of this class, the students valued and developed a greater understanding of sophisticated mathematical ideas and this was highlighted by the motivating and knowledgeable applause that followed a mathematically significant event within the group.

Learning and interaction

The following two sessions described are from Weeks 11 and 12 and are towards the end of the teaching sessions. They demonstrate how students can build an algebraic perspective of problem solving. The focus of the prior lesson was to develop a broader understanding of equivalence, in particular the thinking required to manipulate both sides of an equation. At this stage some of the students were using shortened forms of recording and in some respects their own symbolic representations mirrored the formal algebraic symbolism encountered in secondary school. Keiran (2007) describes this as a *generational* activity where students actively create representation of situations, properties, patterns and relations and many of the symbolic meanings children assigned to their thinking can be viewed as algebraic.

Whole class problem

The following problem was given to the whole class.

> *You are given a balance scale, a lump of clay, a 50 gram weight and a 20 gram weight. Describe how you would use these materials to produce a 15 gram lump of clay.*

The thinking described by Thomas is indicative of many students in the class. He demonstrates an understanding of working on both sides of an equation and understood the relationship between the weights and the clay.

> *Thomas:* Here's what I am thinking. If you've got a 50 gram weight and a 20 gram weight, this side is 30 less than the other. Okay, so you get a lump of clay and put it on there and if it balances out then that is thirty and then you half and you get your 15 grams.

This introductory question built a particular way of thinking that emphasised an interpretation of equivalence based on a balance scale metaphor. The idea that for every mathematical action there is a reaction provides a powerful basis for solving problems using an algebraic perspective. This understanding was then carried through to the next series of problems where the relationship could be expressed as two equations.

Group questions and class discussion

Once each group had presented their explanation they were given a choice of problems to solve. Each group could decide which problem they wished to solve and were encouraged to use their own solution process. While many of the children still used counters and diagrams, a number were now using their own shortened symbolic forms. Sarah's group decided to solve the problem:

One block of weight A and one block of weight B weigh 90 kilograms. Two blocks of weight A and one block of weight B weigh 115 kilograms. How much do three blocks of weight A and one block of weight B weigh?

Liam's group, however, chose the following problem:

At the Flourish and Botts bookstore the first Harry Potter book and the second Harry Potter book together cost $45. Two copies of the first Harry Potter book and three copies of the second Harry Potter book costs a total of $125. At this bookstore how much is the first Harry Potter book?

Both Sara (Figure 1) and Liam's (Figure 2) explanations highlight the significant value of identifying the mathematical relationships between the two unknowns. Both of them were able to use a system of equations to solve the problems. They were able to write this symbolically and their explanation confirms this understanding. Furthermore, after Liam had completed his explanation, Sarah, referring to Liam's example, commented that her problem "is exactly the same as the one we did before". Sarah's statement showed how she acknowledged the problems to be structurally similar even though the content and context were different. Her mathematical focus was not the specific answer to the problem but how both problems could be interpreted in structural terms. An important aspect of algebraic thinking is the ability to consider the interrelationships and generalisation of problem situations and if these generalisations are understood students' mathematical abilities can flourish.

Sarah: Because weight A and B are 90 kilograms, there's two A's and B together there and they weigh 115 kilograms. So you take away the 90 away from 115. It equals 25 kilos. So 1A is 25 kilos and 1B is 65 kilograms.

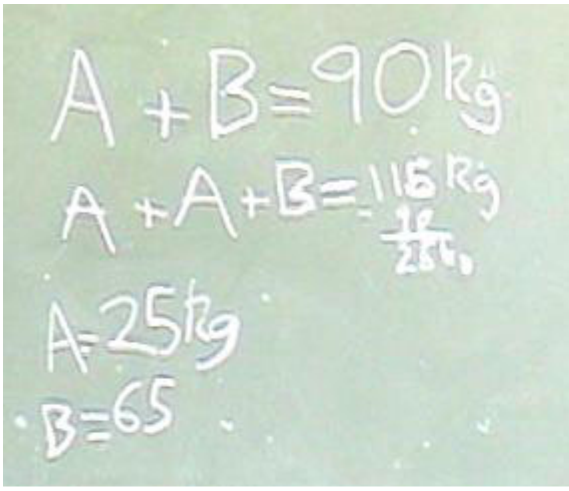

Figure 1. Sarah's explanation to the class.

Liam's explanation follows.

Liam: What we did was 45 double equals 90 so that means that those two together equal 90 (circles 1 and 2). That one is 45 and that one is 45 which is 90 and the one left over is 35 (writes 2 = 35) and that means 45 take-away 35 which means 1 equals 10.

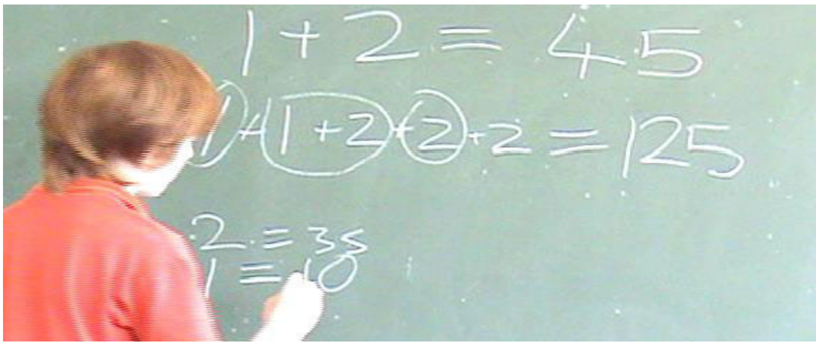

Figure 2. Liam's explanation using 1 and 2 as his symbols for the books.

Holly and Amelia were having difficulty with this problem and they were asked to reflect on how they solved addition and subtraction problems involving unlike common fractions.

> *At the local sports store, all tennis balls are sold at one price and netballs are sold at another price. If three netballs and two tennis balls are sold for $47.00, while two netballs and three tennis balls are sold for $38.00, what is the cost of a single tennis ball?*

Holly explained how she used a factorisation method when both common fractions were unlike and showed an example to Amelia, who through-out the prior lessons had demonstrated an increased awareness and recognition of the mathematical relationships within the problems. The two students then set about solving the problem (Figure 3) and referring to the two netballs and three tennis balls Amelia explained to Holly that the relationship would be maintained if the balls were "increased by a factor of three".

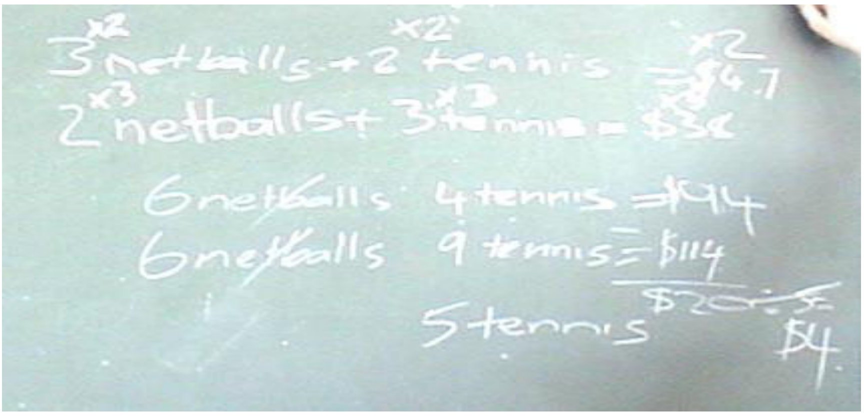

Figure 3. Holly and Amelia's explanation to the class.

In the following lesson and building on from Sarah's, Liam's and Holly's explanation, Dougal's group (Figures 4 and 5) developed a solution using counters and two calculators, whereas Emma's group (Figure 6) showed their thinking using a very detailed diagram for the following problem:

> *At an art store, brushes have one price and pencils have another. Eight brushes and three pens cost $7.10. But six brushes and three pens cost $5.70. How much does one pen cost?*

At this point in time both Dougal and Emma's groups did not understand the factorisation process outlined by Holly and Amelia. However, both had developed an understanding of how to subtract like terms in order to isolate one of the variables. In analysing their interpretation of the problems, the use of the digits 1 and 2 by Liam, Dougal's counters and Emma's diagram of the brushes and pens replaces the conventions associated with using x and y to represent the two variables yet the thinking and to a certain degree the mathematics mirrors a more formal symbolic representation. In developing algebraic thinking these students were capable of developing their own solutions. It must be emphasised that the students' symbolism was not forced upon them, but reflected their own thinking. While it is tempting to move as soon as possible to a formal, symbolic approach as the basis of school algebra, this move may lessen the significance and power of algebra to many learners. The opportunity to be grasped is one that develops a general way of solving problems that allows students the freedom to internalise their thinking and builds an understanding of this symbolism.

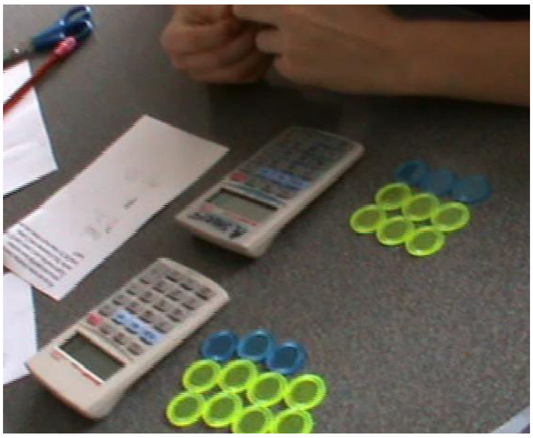

Figure 4. Dougal using two calculator and counters to complete the problem.

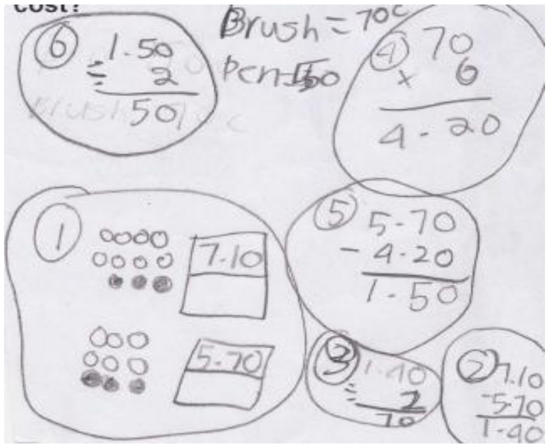

Figure 5. Dougal's written explanation of the same problem.

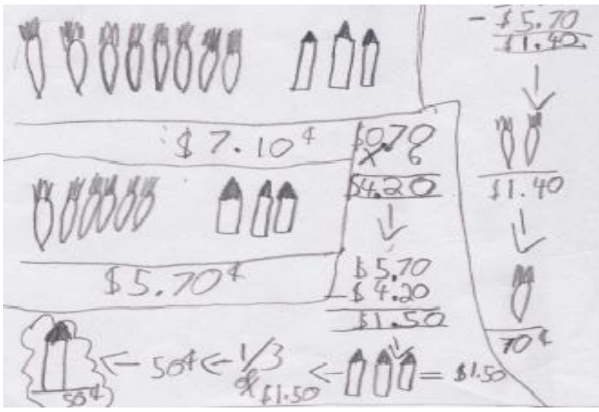

Figure 6. Emma's diagram to solve the problem.

Discussion and conclusions

The current outcomes of this research project indicate that a problem solving approach that develops algebraic thinking and provides students with the foundations in which to reason algebraically. The foundations of the approach are based on facilitating and encouraging students to represent and solve structurally related problems in a variety of ways and giving them opportunities to articulate and generalise their solutions. As a student's generalised and relational thinking develops their initial verbal descriptions give way to more mathematically based explanations, preparing them for the more concise, symbolic arguments that will eventually develop into the formal algebra used in further mathematics. In particular, students can be helped to construct algebraic notation in a meaningful way through their representations using materials, diagrams, models, tables and graphs in their search for patterns and generalisations. This approach empowers a way of thinking about mathematics that can offer students a more meaningful conceptualisation of algebra. By developing algebraic thinking using a problem solving approach, students develop a way of thinking that builds from their own mathematical understanding and provides an entry point into more sophisticated mathematics.

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