VALUE OF WRITTEN REFLECTIONS IN UNDERSTANDING STUDENT THINKING: THE CASE OF INCORRECT SIMPLIFICATION OF A RATIONAL EXPRESSION

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Encouraging students to articulate their thinking when doing mathematics is a means by which teachers ascertain understanding. Reported here are the results from a content analysis of the written reflections of 67 undergraduate students who incorrectly simplified a rational expression. Although asked to write about the thinking that led them to their solutions, most did not. Instead, they recounted what they had done or had not done. Of those who did write about their thinking, most wrote of their confusion or uncertainty; only a few provided a rationale for the procedures they used. Nevertheless, insights into student thinking were gleaned.

Reflection, silent or articulated, undertaken individually or with others, scaffolded or unaided, has a place in the teaching and learning of mathematics. Carpenter and Lehrer (1999, p. 22) state that "reflection involves the conscious examination of one's own actions and thoughts". In the cognitive science literature, reflection has been described as a metacognitive activity. Sjuts (1999) describes metacognition as "knowing and thinking about one's own cognitive system as well as the ability to control and check this system" (p.76). He explains that while reflection can be seen as a metacognitive process, the subject of the reflection involves cognitive processes, such as learning, remembering, understanding, thinking, and knowing.

Carpenter and Lehrer (1999) argue that communication itself can be a reflective act:

> Articulation involves the communication of one's knowledge, either verbally, in writing or through some other means like pictures, diagrams, or models. Articulation requires reflection in that it involves lifting out the critical ideas of an activity so that the essence of the activity can be communicated…. in fact, articulation can be thought of as a public form of reflection. (p. 22)

The benefits of incorporating one form of articulation, written reflection activities, into student learning experiences have been documented for both the school context (Goldsby & Cozza, 2002; Lim & Pugalee, 2004) and the university context (Borasi & Rose, 1989; Parnell & Statham, 2007). Learning benefits have been found in both the cognitive and affective domains. Written reflection can improve students' problem solving, mathematical content knowledge, and understanding. It can also provide therapeutic value. For the teachers, student written reflections can inform their pedagogy. It may provide explanatory data about student misconceptions that shed light on worked solutions and verbal responses. Although Payne and Squibb (1990, p. 445) argue that "important insights into the nature of cognitive skill and its acquisition can be gained by examining errors", making inferences from worked solutions alone has limitations.

An area of mathematics in which student errors and, to a lesser extent, student thinking has been probed is the manipulation of rational expressions. For example, research into students' struggle with simplifying rational expressions, also referred to as algebraic fractions, has had a long history (Grossman, 1924; Guzmán et al., 2010; Storer, 1956). In 1924, Grossman wrote

> Every teacher of experience knows that a great many of his algebra pupils all the way from the first year in high school up to college continue with almost comical regularity to make strange mistakes in the subject of "cancellation" in fractions—mistakes that show clearly that the essence of the matter has escaped them. (1924, p. 104)

Almost ninety years later, there exists an extensive literature that classifies the "strange mistakes" students make in simplifying rational expressions, theorises the thinking that may be causing the errors, and makes recommendations for pedagogy. Yet, students at school and in higher education continue to make errors when simplifying rational expressions. The research reported here adds to this body of knowledge in two ways.

The paper explores the merit of post-solution written reflection, a form of "reflection-on-action" (Schön, 1987, p. 27), for collecting explanatory data on student thinking on this topic amongst undergraduate students. This method of generating explanatory data has been rarely used in this context. The literature suggests that spoken reflection through interviews has been the primary means by which researchers have explored student thinking when working with rational expressions (Guzmán, Kieran, & Martínez, 2010; Nishizawa, Matsui, & Yoshioka, 2002). Secondly, undergraduate students' understanding of rational expressions does not appear to have attracted the same interest as that of school students.

The research reported here is part of a larger study (Ruhl, 2011) investigating student learning, in particular student errors, in the algebraic component of an undergraduate preparatory mathematics subject at an Australian university. The study analysed three sets of data, namely, the worked solutions to a test students sat upon completion of the algebraic unit, the confidence levels they expressed for each question of the test, and the written reflections on the questions they answered incorrectly. The test, the reflection activity, and the preparation leading up to both were part of the teaching and learning experience of all students in the cohort.

This paper focuses on the written reflections that students who volunteered for the study generated for one question in the test. The question asked students to simplify a rational expression in one variable in which the denominator was already factorised. The solution required factorising the binomial expression in the numerator prior to cancelling the one factor common to the numerator and the denominator.

The question was selected because of the high error rate (86% of study participants simplified incorrectly) and the high level of false confidence. Of those who indicated "I am confident I am right", 94% were wrong. Similarly of those who chose 'I am fairly confident I am right', 82% of the responses were incorrect.

Method

An algebra test of twenty questions was administered to a cohort of students enrolled in an undergraduate preparatory mathematics subject at university. The subject is equivalent to a secondary school mathematics subject that prepares students for entry into disciplines at the tertiary level where knowledge of calculus is required (such as engineering, or the natural sciences). A range of students enrol in the subject; some have not satisfied mathematics prerequisites on entry to the university, while others are enrolled in degree programs that have no mathematics prerequisite for entry and are required to study this level of mathematics during their degree.

Students sat for the test after having completed the five week long algebra component which comprised approximately the middle third of the subject. It is assumed that students enrolled in this subject do not have any prior algebraic knowledge.

The students had sat for a similar test at the commencement of the algebra course, the results of which had been used for teaching purposes. That test, which had also been administered to other cohorts, served as a pilot to the final modified test.

The test, taken under formal exam conditions, was worth 15% of the total assessment. Students were directed to show all their working for each question attempted.

Ten days after sitting the test, in a 50 minute lecture timeslot, the marked papers were returned to the students. As well as providing a mark, the examiners highlighted for most questions the parts of the responses where the errors had occurred. A set of written solutions for the test questions was also distributed to the students.

Upon receiving their papers, students were invited to write reflections for the questions they had answered incorrectly. Most students spent 30 to 40 minutes on the task writing on average more than 10 reflections.

The reflection task was scaffolded. In addition to the cognitive prompts of errors being highlighted and the provision of worked solutions, there was also the metacognitive prompt asking students to recall the thinking they experienced at the time of responding to the question. The directions given orally and in writing for the written reflective task included the following:

1. If you have an error highlighted in yellow, compare your answer to the worked solution. Note that not all errors are highlighted.
2. Describe the mathematical thinking you were doing that led you to respond to the question in the way you did.

Students had been encouraged in the intervening tutorials and via email to attend the written reflection session. The benefit stressed was that reflection would help maximise their learning from the test in preparation for their final exam. There was also the incentive of gaining up to 2.5% bonus marks for having written reflections.

Experience in writing reflections on their solutions had been included in the five tutorials leading up to the test. Students experienced a range of reflection activities that included individual and group tasks and oral and written tasks. The ongoing constraint the tutor encountered was the lack of time to develop reflection skills; much of the tutorial time involved re-teaching of the mathematics presented in lectures. Students had little or no experience articulating their mathematical thinking and the reflection tasks in the tutorials met with some resistance.

The target question for which the analysis of the reflections is reported in this paper was written as follows:

Simplify the following rational expression completely.

$$\frac{b^3 + 6b}{3b}$$

The solution for the target question given to the students is reproduced in Figure 1.

Q6	Simplify the following rational expression completely. $\frac{b^3 + 6b}{3b}$	Common factors can't be cancelled, unless the numerator and denominator is factorised first.
	$= \frac{b(b^2 + 6)}{3b}$	1. Factorise the numerator.
	$= \frac{\cancel{b}(b^2 + 6)}{3\cancel{b}}$ $= \frac{b^2 + 6}{3}$	2. Cancel the common factor of b.

Figure 1. Solution for the simplification of the rational expression.

Of the 160 students enrolled in the subject, 151 students had volunteered for the study; of these, 133 sat the test and had provided a response to the target question. One hundred and fifteen of the 133 students (86%) produced incorrect responses to the question; of these, 68 (51%) wrote reflections regarding their response. One reflection was unusable, leaving 67 for analysis.

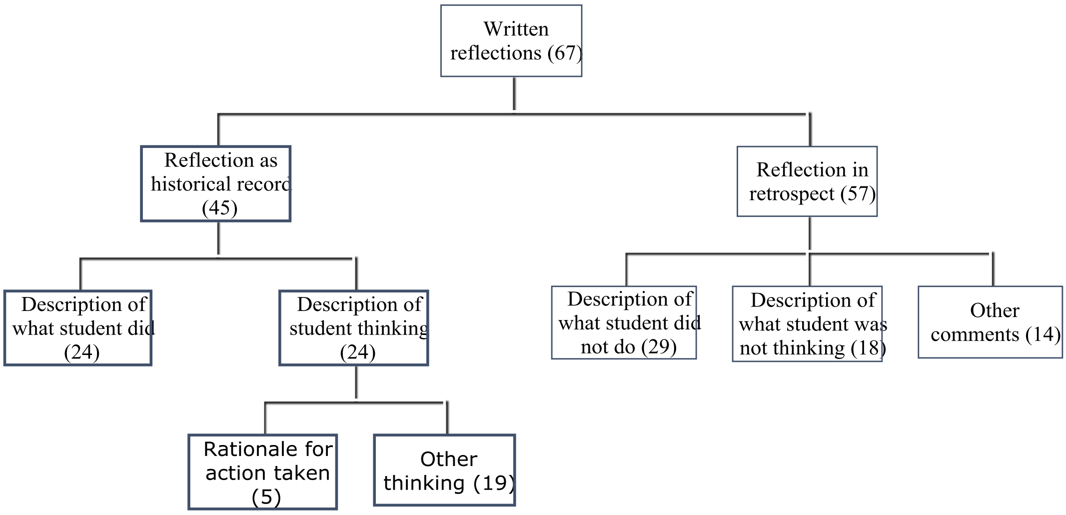
The written reflections were subsequently typed, coded, and recorded using the Nvivo 9 qualitative data processing software. Content analysis was used to categorise or code the reflections in a two part process. On Patton's (2002) inductive-deductive continuum along which he places qualitative research methodologies, the analysis used in this study would best be described as inductive but with some tentative pre-existing conceptual guidelines. Using the analogy of a category as being a "bin" in which data are placed, Miles and Huberman (1985) note that "any researcher no matter how inductive in approach knows which bin to start with and what their general contents are likely to be"(p. 28).

After the three researchers had read and reread the reflections, the first "bin" or category became "Did the students do the task required i.e., did they describe the 'mathematical thinking' that had led them to their response, and if not, what did they do?" This led to categorising the reflections according to type of reflection. The second phase of analysis focussed on how the reflections contributed to understanding the student thinking that led to the specific errors evident in their solutions. The errors in the responses had been categorised prior to analysing the reflections (Ruhl, 2011). This second process required analysing the reflections with reference to the student worked solutions.

The analysis of the data took place the semester following the delivery of the subject. As the main researcher (Ruhl) was the sole tutor for the subject, ethics required that the names of the students who had volunteered for the study would be available only after the results for the subject were released.

Results and discussion

The results of the first phase of analysis in which the reflections were coded according to type are summarised in Figure 2. The number attached to each category indicates the number of reflections coded in that category. The first categorisation used a temporal dimension distinguishing between the reflections that were a historical record, for example, "Tried to cancel the *b* with top and bottom", from those that were written from the perspective of hindsight or in retrospect, for example, "I should have factorised". Of the 67 reflections coded, 35 were coded in both categories.



Note: Some reflections were coded in more than one category.

Figure 2. Reflections coded according to type.

The second level of coding subcategorised each of the two sets in terms of whether the reflections referred to "doing" (see examples above) or "thinking". Two "thinking" examples were "Thought I could cancel because the base letters were the same" and "I forgot to factorise". The "reflections in retrospect" required a third category called "Other comments" which included reflections such as "A lot more study needed perhaps".

The third level of coding further categorised the five categories from the second level of coding. However, the figure includes only the coding done for the "description of student thinking" category for the reflections that recorded what students believed they were thinking at the time of simplifying the rational expression. The two most important findings from this phase were the large number of reflections that focussed on "doing" rather than "thinking" and the very small number of reflections that were a rationale for simplifying the rational expression in the chosen manner. The reflection types for each error type identified in students' solutions are shown in Table 1.

Table 1. Summary of reflection type for error type in student solutions.

Error			Reflection		
Type	Example	No [1]	Description of what student did[2] (No)	Description of student thinking (No)	
				Rationale	Other
Simple cancellation	$\frac{b^3 + \cancel{6b}2}{\cancel{3b}} \Rightarrow b^3 + 2$	30	18	3	9
Cancellation by subtraction of like terms	$\frac{b^3 + \cancel{6b}3b}{\cancel{3b}} \Rightarrow b^3 + 3b$	6	3	1	1
Cancellation by division of coefficients retaining the variable	$\frac{b^3 + \cancel{6}2b}{\cancel{3b}} \Rightarrow b^3 + 2b$	2	0	1	1
Conjoining error	$\frac{b^3 + 6b}{3b} \Rightarrow \frac{6b^4}{3b}$	1	0		1
Other (e.g., computational errors, and uncodeable solutions)	$\frac{b^3 + 6b}{3b} \Rightarrow \frac{\cancel{3b}(b^2 + 2)}{\cancel{3b}}$	5	1		5
Working absent		2	2	0	2
Total		46	24	5	19

Note 1. The total is 46 rather than 45 (Figure 2.) because one student made two errors.
Note 2. Some reflections were categorised in more than one category.

Of the 24 reflections that recorded what students did in simplifying the rational expression, 15 referred to "cancelling" (14) or "eliminating" (1). Thirteen of the 15 were reflections for solutions where students had made a "simple cancellation error". A breakdown of these 13 reflections showed that

- 5 referred to cancelling or eliminating "common factors"
- 2 referred to cancelling "variables"
- 2 referred to cancelling "numbers"
- 3 made no reference to what was cancelled
- 1 cancelled "some properties of the expression"

It is possible to infer from these reflections that students realised that simplification of the rational expression requires cancellation of "something" common to the numerator and the denominator with the "something" being described in various ways. However, their understanding of what constitutes a common factor appears to be that it is either a number or a variable that is found in a term in the numerator and in a term in the denominator. Hence none saw the need to factorise the numerator.

Not all cancellation errors however, seem to be associated with failing to factorise. The student script reproduced in Figure 3, for example, indicates that the student knew to factorise the numerator. In this instance, the component of the written reflection analysed stating "took 1b from 3b" reinforces the categorization of this error as a "cancellation by subtraction of like terms" error.

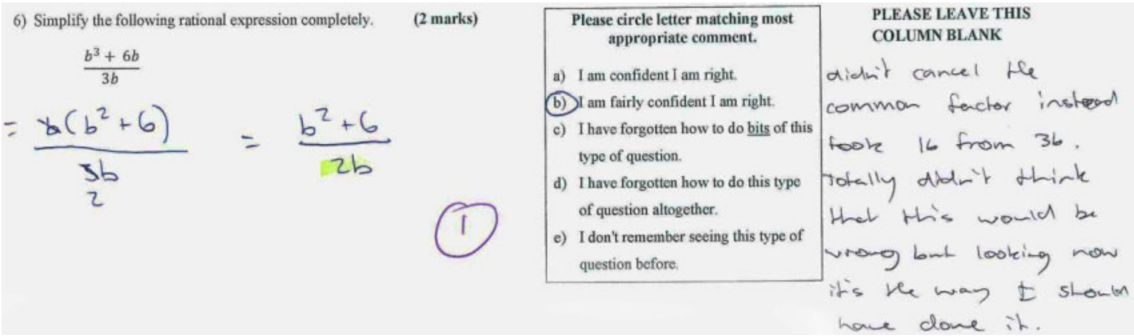


Figure 3. Student script for a cancellation by subtraction of like terms error.

Of the 19 thinking reflections categorised as “other thinking”, all, with one exception, expressed confusion or uncertainty. Examples include, “I got confused” and “Was unsure how to do it”. The exception was that of a student who had performed a “simple cancellation” error. Her script is worth commenting on (Figure 4) because, unlike the previous case, the reflection does not appear consistent with the worked solution.

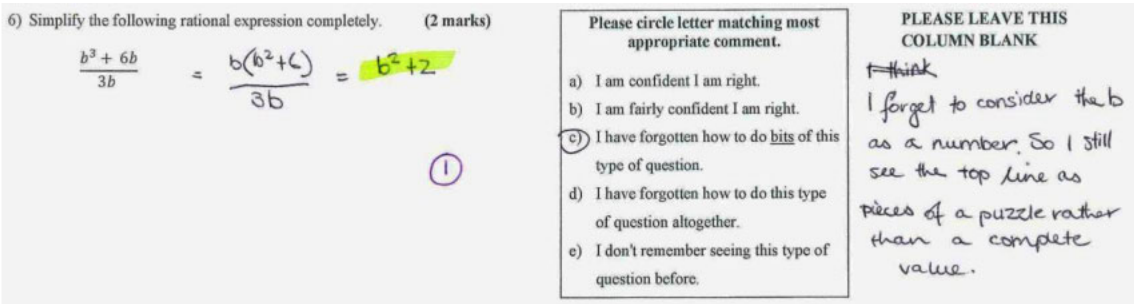


Figure 4. Student script for a simple cancellation error.

Apart from the “simple cancellation” error, the worked solution appears to indicate that the student appreciates the need to factorise the numerator before cancelling. The common factor *b* also appears to have been cancelled successfully. Yet the reflection indicates that perhaps the student does not understand why she manipulated the expression in the way she did. She states, “I still see the top line as pieces of a puzzle rather than a complete value”. This image suggests that the student sees the elements of the numerator as discrete pieces that can be lifted and discarded when their match is found on the denominator.

Finally, the category that explicitly described the thinking that led students to their solutions contained five reflections. These are reproduced below. Four of the set provide the opportunity to see that conceptually similar reflections need not mean similarly worked solutions.

> “I simplified the 6 and the 3 by 3 because they were both common factors of each number.”
> “I believed you could cancel if the numerator and denominator had same letters/symbols.”
> “I was thinking that because 6b and 3b are like terms I could just cancel.”
> “Thought that I could cancel because the base ‘letters’ were the same.”
> “I tried to divide by 3b, because the question said to simplify, I looked for like terms. 6b÷3b=2b.”

The first three reflections were written by students who made the "simple cancellation" error; the fourth was written by someone who made the "cancellation by subtraction of like terms" error and the fifth was by a student who made the "cancellation error involving the division of coefficients while retaining the variable". Apart from the first reflection, the remaining four seem to share an understanding that cancellation in a rational expression involves the cancellation of "like terms". Notwithstanding the common ground amongst the reflections, the corresponding solutions displayed different errors and different end results. The result corresponding to the second and third reflections were both b^3+2; the result corresponding to the fourth reflection was b^3+3b; and the result from simplifying the rational expression that corresponded to the last reflection was b^3+2b.

The results produced in this study were influenced by a number of conditions that may have limited the quality and the quantity of the reflections. Firstly, time constraints meant that learning how to reflect mathematically had been limited. Secondly, the time allocated to writing the reflections was limited; students may have sacrificed depth for breadth. Thirdly, the scaffolding, in particular, the worked solutions, may have strongly influenced the nature of the responses and provides at least a possible partial explanation for the large number of reflections that focussed on the teacher's solution as their point of reference.

Conclusions and implications

In conclusion, the findings from this study have implications for pedagogy in the algebra unit of undergraduate preparatory mathematics subjects. Using written reflections to generate explanatory data about student thinking has the benefit of accessing a large number of students in a time efficient way. However, it does not allow for the teacher/researcher prompts that dialogue offers which can lead to richer reflections. Notwithstanding this limitation, the study showed that written reflection provides insights into the student thinking, including its contradictions and anomalies, that contributes to incorrectly simplifying rational expressions as well as revealing the difficulty that students have with writing about their thinking.

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