

# Teaching Percentage as a Multiplicative Relationship

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The theory of Teaching for Abstraction was used to construct a unit for teaching percentages in Grade 6. The unit focused on percentage as a multiplicative relationship and put more emphasis on why percentages are used than on how they are calculated. Analysis of the data collected showed that many students did learn to apply percentages meaningfully, although the level of achievement was not as high as expected. It was found that the Teaching for Abstraction approach provides many challenges for students and teachers, but is worthy of further research.

The role of abstraction and generalisation in mathematics learning has received substantial attention in the last ten years and was afforded its own research forum at the 26th international conference of PME (Boero et al., 2002). Mitchelmore & White (2004) emphasised the importance of empirical abstraction in mathematics learning, focusing on an abstract concept as “the end-product of ... an activity by which we become aware of similarities ... among our experiences” (Skemp, 1986, p. 21). This view of abstraction leads to an associated theory for teaching early mathematical concepts called *Teaching for Abstraction* (Mitchelmore & White, 2000a), in which students engage in:

- *familiarising* themselves with the structure of a variety of relevant contexts,
- *recognising* the similarities between these different contexts,
- *reifying* the similarities to form a general concept, and then
- *applying* the concept in new situations.

Much of the theory has been developed from investigations into young children’s understanding of the angle concept (Mitchelmore & White, 2000b), but also into mathematical concepts involving rates of change (White & Mitchelmore, 1996) and decimals (Mitchelmore, 2002). The project reported in this paper begins to expand this research to other areas of the mathematics curriculum.

## *Multiplicative Relationships*

Research has identified a number of mathematical concepts which students find particularly difficult to learn. The well known study by Hart (1982) categorized many of these and one which stood out was ratio and proportion. A cursory look at the school mathematics curriculum demonstrates that multiplicative relationships (proportionality) underpin almost all number-related concepts (e.g., fractions, percentages, ratio, rates, similarity, trigonometry, rates of change). Could it be that students’ poor performance on tasks involving proportionality is due to the fact that curriculum materials rarely highlight the multiplicative nature of these concepts? Is the problem that students do not recognise this underlying similarity and hence do not develop a general concept of proportionality?

Research has shown that, when confronted with a situation requiring proportional reasoning, students regularly resort to inappropriate additive strategies. For example, one item in Hart’s study asked students to enlarge a 5 cm × 3 cm rectangle so that the new length was 12 cm. Less than 20% of a sample of 2257 students aged 13-15 gave the correct answer of 7.2 cm as the new width. A common error was 10 cm, found by adding the

additional length (7 cm) to the original width (3 cm) rather than using a multiplicative relationship ( $\frac{12}{5} \times 3$  cm). Misailidou and Williams (2002) recently verified Hart's findings, but also found that student teachers were largely ignorant of students' tendency to use additive strategies.

Percentage is a multiplicative relationship which causes students particular difficulties. A detailed analysis of the concept by Parker and Leinhardt (1995) also highlighted the multiplicative-additive confusion. They demonstrated that the concise, abstract language of percentages often uses misleading additive terminology with a multiplicative meaning. Their review of research into teaching percentages showed that a method which linked the development of calculation procedures closely to percentage contexts was likely to be more effective than one in which procedures were taught abstractly and then applied.

More recent research has seen an increasing emphasis on the relationship between different multiplicative topics (Litwiller & Bright, 2002). In particular, Moss and Case (1999) proposed teaching percentages in Grade 4 and then using students' knowledge of percentages as a basis for teaching decimals and fractions. They found that students taught by this method showed a deeper understanding of rational numbers than those in a control group. More recently, Moss and Caswell (2004) have extended this approach up to Grade 8 and included a greater focus on proportional reasoning.

The topic of percentages was therefore chosen as a most appropriate one for research into the further applicability of Teaching for Abstraction, with the aim of later extending the research into other multiplicative contexts. In the experimental teaching unit described in this paper, students investigated a variety of situations where it was or was not appropriate to use percentages, and were led to become aware of what it is that links all the percentage situations. It put much more emphasis on *why* percentages are used than the current curriculum, which tends to emphasise percentages as numbers (e.g.,  $5\% = \frac{5}{100} = 0.05$ ) and how to calculate with them (e.g.,  $5\% \text{ of } \$250 = \$12.50$ ). This is not to say that calculation was seen as unimportant—the learning activities also aimed to develop computational facility.

## Method

### *Participants*

Six teachers and their students from one public and one private school in Sydney participated in the study. In one school, the classes were unstreamed and contained only Year 6 students. In the other, Year 5 and Year 6 students were combined and one class was identified as containing more gifted students.

### *The Teaching Unit*

The four phases of the theoretical framework for Teaching for Abstraction were used in planning the activities for the experimental unit as follows.

- *Familiarising*: Students explored individual contexts assumed to be familiar to them, investigating concepts such as fair sharing and taxation. Only simple percentages were used (50%, 10%).
- *Recognising*: Activities required students to compare and contrast the use of percentages in specific contexts. Calculations for more complex percentages were based on recognising the pattern for calculating with 10%.

- *Reifying*: Students were asked to make and explain generalisations based on the similarities found in the Recognising phase. These generalisations were then used to carry out abstract (i.e., decontextualised) percentage calculations.
- *Application*: Students created their own problems involving the application and calculation of percentages.

The resulting lesson topics are shown in Table 1.

Table 1  
*Topics for Percentage Lessons*

Topic	Main student activity
1. Robot Rivals	Compare two different robots to an original (one has been enlarged by 50%, the other by 50 mm). Investigate which enlargement maintains the shape of the robot.
2. Calculating Percentages	Extend previous experience of percentages, assumed to be limited to 10% and 50% of 100 objects, to simple percentages of any number of objects.
3. Fair Sharing	Compare different ways the GST could be charged and decide on fair ways.
4. What is the best way?	Investigate problems involving different comparisons and decide the best way to solve these problems.
5. Who wins?	Play a game involving percentages of numbers of different sizes, in which they can use multiplicative or additive strategies. Decide which strategy gives the best chance of winning.
6. Discounts & Combinations	Investigate discounts and the effects of combining two discounts.
7. Comparisons	Make percentage comparisons between parts of two different wholes.
8. Summary	Summarise all the main ideas and skills learnt in this unit.

### *Teacher Preparation*

The six teachers met with the authors in two one-day sessions, once before teaching the unit and once afterwards.

At the first session, teachers were provided with a folder containing a brief introduction to the philosophy of the unit, outlines of the eight lessons, and instructions on what they were expected to do. The day started with a general discussion of what is important in teaching percentages, after which the researchers outlined their approach to learning and summarised the content of the experimental unit. The teachers then worked through each lesson, anticipating how their students would react and making suggestions for improvements. Each teacher was asked to teach the unit over the following two weeks, and to identify a target group of five students in their class for closer study.

In the second session, teachers gave detailed feedback on their experiences and evaluated the unit as a whole.

### *Data Collection*

A research assistant was present at the two teachers' days and visited schools regularly to assess the target students' understanding of percentages before and after the teaching, to observe lessons, and to interview teachers and their target students after each lesson. This procedure generated multiple sources of data for evaluating the unit:

- Pre- and post-assessments of the target students
- Interviews with target students
- All worksheets completed by the 30 target students
- Observations of two lessons per teacher and subsequent interviews with the teacher
- Teachers' evaluations of each lesson and of the unit

Student understanding of percentages was assessed in a 15-minute interview in which students were asked to judge whether or how to apply percentages to four hypothetical situations. The following item was typical:

Two basketball players compare their shooting averages from the free throw line. The first player has scored 20 goals from 40 shots. The second player has scored 25 goals from 50 shots. Which player is the better shooter?

Responses to all items were scored correct only if they included an explanation which referred to an appropriate multiplicative relationship, usually a percentage or a ratio.

### **Results**

In discussing the results, we shall focus on the student learning that took place. Teacher learning will be reported in a separate paper.

#### *Quantitative Data*

Since the samples in this study were not chosen at random, tests of statistical significance are, strictly speaking, not valid. We shall nevertheless use them as a guide to data interpretation.

Figure 1 shows the mean total scores on the percentages assessment instrument in each school before and after the teaching. It appears from Figure 1 that School 1 was lagging behind School 2 before the teaching but had caught up by the end. However, repeated measures ANOVA (Table 2) showed that there was no significant difference between the schools and no significant interaction between them and the assessments ( $p > 0.10$ ). This finding indicates that the difference between the schools might well have been due to sampling errors, and allows us to pool the data for further analysis.

Overall, the mean total score increased from 28% to 56% from the pre-assessment to the post-assessment, a highly significant difference ( $p < 0.001$ ). For example, on the item quoted above 57% of the students chose either the first or the second player on the pre-assessment whereas only 33% gave this response on the post-assessment. Such students usually used an additive argument: Tai chose the first player and explained that *they both got half their shots in, but the first player did it with less shots*. Ashley chose the second player *because he got 25 and  $50 - 25 = 25$  and  $40 - 20 = 20$  so 25 is more*. Students who gave a response like *Neither, they are both equal* were scored correct on the assumption that they had used a multiplicative comparison.

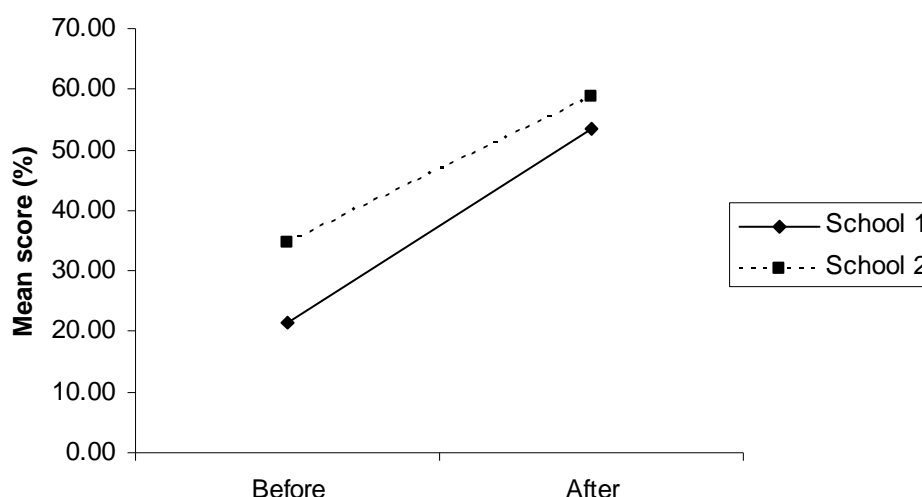


Figure 1. Mean total scores before and after teaching, by school

Table 2  
Analysis of variance of percentage assessment scores

Source of variation	SS	df	MS	F	P-value
Assessments	29.40	1	29.40	15.51	0.00
Schools	3.27	1	3.27	1.72	0.19
Interaction	0.60	1	0.60	0.32	0.58
Students	106.13	56	1.90		

The quantitative data thus indicate that the experimental lessons supported learning of the desired outcomes. But the average score was still only just over 50%, and six students (20% of the sample) who scored poorly on the pre-assessment made no gains at all.

### Qualitative Data

Instead of attempting to summarise the learning that did and did not take place in each lesson, we shall attempt a more global summary of the qualitative data under three headings: use of familiar situations, explaining and generalising, and learning calculation skills.

#### Use of Familiar Situations

A feature of percentage is its frequent use in familiar, everyday situations (Parker & Leinhardt, 1995). The teachers spontaneously identified this feature in the pre-teaching workshop and welcomed its reflection in the teaching materials. Students, for their part, recognised the situations presented, were willing to share their experience and contribute their knowledge, and became deeply engaged in analysing them and arguing about how best to make comparisons. For example, Molly reported after the first lesson: *I liked doing things and not just writing maths and numbers.*

However, some of the situations were clearly unfamiliar to students—for example, the idea of paying taxes in Lesson 3. We felt this lesson would link well with discussions of tax in social studies (Human Society and Its Environment—HISE). However, many

students seemed to think either that all taxes are bad and therefore unfair or that all taxes are fair because the government needs the money to pay for schools, hospitals, and other services. As a result, many students found this lesson confusing and missed the point of the mathematical discussion.

Another aspect of realistic situations is that they are seldom black-and-white. For example, Lesson 4 included this question:

Poor Alfred lost 50% of his money. Circle which you think would be worse for Alfred. Originally having:     \$1     \$10     \$50     \$100     original amount makes no difference

This question was intended to highlight the fact that percentages are meaningless in themselves but must always be interpreted in relation to the context. Several students gave good answers. In one class, Elyse circled \$100 and wrote *Because 50 percent [sic] of 100 is \$50 and that's a lot of money*. Lang circled the “no difference” option and wrote *You don't know how old he is and you don't know the importance of the money*. However, their teacher seemed uncomfortable with the idea that both answers might be right (i.e., contain an element of truth), marked Elyse's answer as wrong, and afterwards remarked that she felt the question was meaningless. Such a reaction, if common, could clearly stifle the development of a critical attitude towards the application of percentages.

### *Explaining and Generalising*

A feature of the experimental unit was the emphasis on making and explaining empirical generalisations. This was clearly difficult for students and teachers alike.

For example, in Lesson 1 students were asked to describe how a robot figure had been changed. Ashley wrote:

Roboton's head width has been increased by 2, Roboton's hip length by 2, Roboton's leg length by 3, Roboton's body length by 4 and Roboton's arm length by 4.

Ashley seemed to have difficulty summarising all these individual changes into one generalisation (Roboton's body dimensions had been increased by 50%).

In the game in Lesson 5, students started with 100 points. They then threw a die and were allowed to choose between adding or subtracting a corresponding number of points or that percentage of their current number of points. Many students could not decide on a good strategy. Even those that devised a winning strategy could not explain why it worked. Elyse said *Over 100 and going up it's better to choose percentages. I don't know why*.

One of the aims of Lesson 6 was to develop the idea that a percentage only has meaning in relation to its base (i.e., you have to know what it is a percentage of). Students compared the result of applying a fixed and then a percentage discount to the result of applying them in the opposite order. Many students obtained the correct answers but again could not explain their results. When asked *Does the price or the discount affect this result?* students became confused and some just answered *yes*. They did not seem to understand what the question was asking.

It seemed from discussions with teachers and students that students were simply not used to being asked for explanations or generalisations. The observer also felt that teachers did not do enough to encourage generalisation. Most lesson plans included a final step asking teachers to develop a generalisation out of the students' activities, but they often left out this step—sometimes because of time pressure.

## *Learning Calculation Skills*

The unit aimed to teach students how to calculate any multiple of 10% by first calculating 10%. This method was intended to allow for quick, mental calculations without complications arising from simplifying fractions. It was introduced in Lesson 2 through the activity of sharing jellybeans into 10 boxes—starting with 100 jellybeans, then 200, then 160. Sarah was clearly successful in learning this skill: *At the beginning I didn't know how to work out 10% and 60% and so on but after a while I worked out about the 10% and multiplying it by 6.* However, it was clear from subsequent lessons that many students had not internalised this method. As a result, they often spent so much time on what were supposed to be quick mental calculations that they were unable to reach the intended generalisations concerning the application of percentages.

There were several reasons why students did not learn the intended calculation method:

- There were not enough examples given in Lesson 2. Several teachers suggested a more gentle development and further reinforcement before the method was used in applications. Another explanation is that teachers did not give students enough freedom to explore and discover the target method through their own activities.
- Students seemed reluctant to find a percentage of any number other than 100. Emma said: *I didn't understand 50% and I still don't get it. I thought you only used 50% when you have 100.* This misconception was not directly addressed in the unit.
- The use of percentages to express an increase or decrease was not familiar. Teachers stated that the idea of a 50% increase (used in Lesson 1) was a “big leap” for some of the weaker students.
- Some teachers taught a mechanical method involving fractions in preference to the mental method presented in the lesson notes. In Lesson 7, for example, given that a 10 g biscuit contains 2 g fat and asked to calculate the amount of fat per 100 g, these teachers expected students to find  $2 \div 10$  and multiply by 100. Such a method would have done nothing to develop the intuition that, if there is 2 g fat in a 10 g biscuit, there would be 20 g in 100 g.
- Many teachers were themselves not confident with percentages. In one lesson, the teacher said quietly to the observer: *I don't know how to work it out, so how can I explain it to them?*

## Conclusion

The Teaching for Abstraction approach incorporated in this study is radically different from what students and teachers are accustomed to, involving analysis of realistic situations, value judgments, student explanations, and making generalisations. Such an approach provides many challenges for teachers, including addressing generalisations, distinguishing contextual from abstract factors, accepting multiple answers, and coping with their own lack of confidence in calculating percentages.

Despite these difficulties, the analysis of the pre- and post-assessments showed that many students did learn to apply percentages meaningfully, even though all students found this idea difficult initially. However, the final level of achievement was not as high as had been expected.

This first attempt to use the Teaching for Abstraction approach for percentages has proven to be similar to our early research on the teaching of angles: The results are encouraging, but point to several deficiencies which need attention. There is a need to explore separate contexts in more detail (the Familiarising phase), perhaps being less

ambitious in terms of student understanding or allowing more time. Calculation skills are another area requiring attention (Recognition): Further skill needs to be developed in each context, without assuming any prior knowledge, and then more emphasis put on identifying the similarities. This would allow more emphasis to be put on explaining when and why percentages “work” (Reification).

Teacher preparation for Teaching for Abstraction also needs reconsideration. In particular, the importance of discussion and the need for students to make and justify their own generalisations should be given more emphasis. Teacher confidence and ability in working with percentages also needs to be strengthened. The implication is that teacher preparation needs to allot more time to content than is currently popular in this country (White, Mitchelmore, Branca & Maxon, 2004). The theory of abstraction not only provides a radically new perspective on how students learn concepts identified as problematic but also leads to a firm basis for developing more effective teaching methods. Further research could result in better learning outcomes right across the mathematics curriculum.

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