A POPPERIAN CONSILIENCE: MODELLING MATHEMATICAL KNOWLEDGE AND UNDERSTANDING

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Goldin (2003) and McDonald, Yanchar, and Osguthorpe (2005) have called for mathematics learning theory that reconciles the chasm between ideologies, and which may advance mathematics teaching and learning practice. This paper discusses the theoretical underpinnings of a recently completed PhD study that draws upon Popper's (1978) three-world model of knowledge as a lens through which to reconsider a variety of learning theories, including Piaget's reflective abstraction. Based upon this consideration of theories, an alternative theoretical framework and complementary operational model was synthesised, the viability of which was demonstrated by its use to analyse the domain of early-number counting, addition and subtraction.

Introduction

An *alternative theoretical framework* has been proposed (Nutchey, 2011) that explicitly differentiates, and is thus able to describe, the knowledge shared in the learning community and each learner's idiosyncratic understanding. This proposition is an attempt to address the perceived challenges of reconciling student-centred, constructivist learning and the state-able, objective structure of mathematics shared by a community of mathematicians. In this paper, literature substantiating this need is first identified, and then key theoretical constructs that inform the proposed alternative theoretical framework are summarised. The proposed theory is complemented by an *operational model*, of which a significant component is a graphical language for describing the organisation of a domain of mathematical knowledge shared by a community. This language is introduced, and then its viability is illustrated by applying it to the analysis and description of one perspective of early-number counting. A broader discussion of the viability and significance of the alternative theoretical framework, operational model and graphical language is then provided.

Background

Various theoretical bases are promoted for the teaching and learning of mathematics. Objectivist theories are often criticised for not taking into consideration the learner's prior experience (Lesh, 1985). Despite these criticisms, objectivist-based practice remains prevalent in many of today's classrooms (Falk & Millar, 2001) and computer-mediated learning environments (McDonald et al., 2005). Constructivist-based reformists argue for the development of experientially-based richly connected schemas of understanding. However, such practice is often stymied by the difficulties encountered when attempting to translate the constructivist theory into classroom practice (Baroody & Dowker, 2003; Scardamalia & Bereiter, 2006; Simon, Tzur, Heinz, & Kinzel, 2004; Steffe, 2004). Some of these difficulties have been attributed to the lack of focus upon the highly structured nature of mathematical domain knowledge (Kirschner, Sweller, & Clark, 2006; Mayer, 2004). The often occurring polarisation of these two theoretical viewpoints has been criticised (Goldin, 2003; McDonald et al., 2005) as having a deleterious effect on mathematical education practice. Instead, such critics argue that each viewpoint has its associated strengths, and that these should be drawn upon to develop effective educational practice which recognises both learner idiosyncrasy and the state-able and thus objective nature of mathematical domain knowledge. Through the development of such a consilience (Goldin, 2003) of learning theory, contemporary mathematics education practice may be advanced.

Popper's (1978) three-world model of knowledge and understanding has been adopted as a lens through which to re-consider the objectivist and constructivist theories. Popper's three-world model permits the explicit differentiation of World 3 knowledge shared in a community from the unique, experience-based World 2 understanding of that knowledge held by each community member. In this model, World 2 understanding mediates between the World 3 knowledge of the community and the individual's physical actions of World 1. That is, if the organisation of mathematical ideas that define the shared World 3 knowledge of some mathematical domain can be described, then the individual learner's idiosyncratic and experientially developed and demonstrated World 2 understanding can then be mapped against this description of knowledge. This differentiation and modelling of both World 3 knowledge and World 2 understanding is at the core of the proposed alternative theoretical framework.

Piaget (1977/2001) proposed reflective abstraction as a process of accommodation by adaptation that is sufficiently powerful to describe a learner's entire conceptual development in mathematics. Five specific processes, or transformations, of reflective abstraction are noted in Piaget's work (Dubinsky, 1991): interiorisation, coordination, encapsulation, generalisation and reversal. Interiorisation involves the internalisation and then re-presentation of some phenomena in a de-contextualised, more abstract way. Coordination involves the composition of two or more existing processes to form a more complex process. In a related way, encapsulation involves the bringing together of what were previously independent parts into a manipulable whole. This whole may represent the abstraction of a commonality between a set of concepts or the abstraction of a detailed process into a single object. Generalisation is the broadening of understanding by the application of existing processes and structures to a wider collection of problem phenomena. Finally, reversal involves the consideration of the differences between concepts and the subsequent abstraction of inverse or 'undoing' relationships.

Reflective abstraction is typically associated with the cognitive processes by which individuals construct idiosyncratic understanding. However, when Popper's three-world model is adopted and World 3 knowledge is defined as the expression and thus sharing of idiosyncratic mental thought and cognitive process (i.e., World 2 understanding), then the description of shared World 3 knowledge will reflect these transformations of reflective abstraction. Thus, reflective abstraction forms the basis of the proposed graphical language for describing mathematical World 3 knowledge.

The graphical language

The graphical language for describing World 3 knowledge is a major element of the proposed operational model that embodies the alternative theoretical framework. The graphical language is used to create *genetic decompositions*, a term borrowed from the work of Dubinsky (1991). A genetic decomposition is a network-like structure of nodes and links, and is in keeping with the notion of schema discussed in constructivist literature. Each node in a genetic decomposition is referred to as a *knowledge object*, and these knowledge objects are linked by one or more *knowledge associations*. Each knowledge association in the genetic decomposition describes some reflective abstraction-based relationship between the knowledge objects involved. In the following sub-sections, these constructs of the graphical language are discussed in greater detail.

Knowledge objects

Knowledge objects form the nodes in the network-like genetic decomposition, and of these there are three different types. At the core of learning in mathematics is the solution of problems, and so one type of knowledge object is the *problem object*. To form solutions to such problems, conceptual knowledge (i.e., principles, facts) may be drawn upon as well as procedural knowledge (i.e., skills and processes). Considering Baroody's (2003) suggestion that to foster adaptive expertise conceptual and procedural knowledge should be integrated together, the second type of knowledge object is the *concept object*. Central to mathematical activity is the use of language to express the problems and concepts of the domain. To this end, the third type of knowledge object is the *representation object*, which is used to identify the different signs and symbols of the domain. The three knowledge object types are denoted in the graphical language using three different icons, as shown in Figure 1.

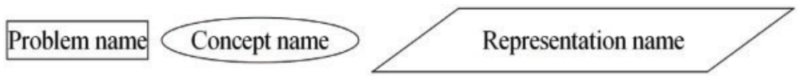

Figure 1. Knowledge objects.

Knowledge associations

To organise the knowledge objects in a genetic decomposition, six different knowledge associations have been derived from Piaget's reflective abstraction: *inheritance*, *aggregation*, *solution*, *inversion*, *formalisation* and *expression*. These associations are discussed in the remainder of this section: the syntax of each in the graphical language is shown in Figure 2, and then each of the associations and their derivation from Piaget's reflective abstraction are summarised.

Inheritance – the *Parent* concept (or problem or representation) is a super-class, of which *Child 1* and *Child 2* are more specific sub-types.

Aggregation – the *Aggregate* concept (or problem or representation) is composed of *Component 1* and *Component 2*.

Solution – the *Problem* can be solved using the coordination of *Concept 1* and *Concept 2*.

Inversion – the *Normal* and *Complement* concepts (or problems) have differences.

Formalisation – the *Formal* representation is a de-contextualised representation compared to the *Informal* representation.

Expression – the *Concept* (or problem) is expressed using the *Representation*.

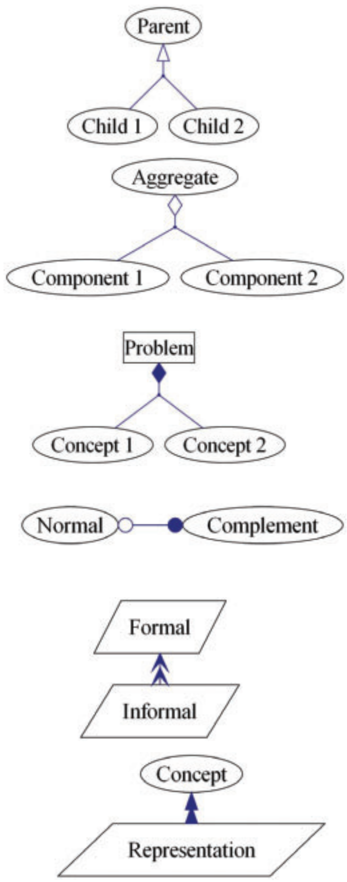

Figure 2. Knowledge associations.

Derived from Piaget's encapsulation, the inheritance association describes either problem, concept or representation objects that share a super-ordinate relationship: The *child* objects are sub-types of the more abstract *parent* object. Inheritance is denoted by an open triangle attached to the parent, from which two or more lines connect the parent to each child.

The aggregation association is derived from Piaget's coordination, and is used to describe the coordination of several *component* parts to form a more complex *aggregate* whole. Aggregation may be applied to either problem, concept or representation objects. The association is denoted by an open diamond attached to the aggregate, from which one or more lines connect the aggregate to each component.

The solution association defines the relationship between a problem and the concepts used to solve it. This association is derived from Piaget's coordination and encapsulation (i.e., the use of two or more concepts in a coordinated manner to solve a problem) as well as generalisation (i.e., since a problem may be solved using several different co-ordinations of concepts, or a set of coordinated concepts may be used to solve a range of different problems). The solution association is denoted using a closed diamond attached to the problem, from which one or more lines connect the problem to each concept.

The inversion association is derived from Piaget's reversal, and is used to describe two knowledge objects (either problems or concepts) that are in some way complementary to each other. The two knowledge objects are referred to as the *normal* and *complement* objects. The association is denoted by a line connecting the two objects that is terminated by open and closed circles.

The formalisation association describes the increasingly abstract signs and symbols used in mathematics. Derived from Piaget's interiorisation, formalisation captures the relative degree of de-contextualisation between two representation objects, which are referred to as the *informal* object and the more de-contextualised *formal* object. The formalisation association is denoted by a line connecting the informal and formal representation objects which is terminated by open arrow-heads.

The expression association describes the various ways by which a problem or concept may be represented, that is, how the signs and symbols of the domain may be used. This association is also derived from Piaget's interiorisation, since when considered in combination with formalisation, the expression association suggests opportunities for interiorisation to occur. The expression association is denoted by a line connecting a representation object to a problem or concept object which is terminated by closed arrowheads.

Application

To demonstrate the viability of the proposed alternative theoretical framework, operational model and graphical language, literature regarding the domain of early-number counting, addition and subtraction was analysed and described (Nutchey, 2011). In this section, a summary of the analysis and description of Gelman and Gallistel's (1978) work on children's counting is provided to demonstrate the use of the proposed graphical language.

Gelman and Gallistel (1978) theorised that a child's ability to count is based on the coordination of five principles: the one-one principle, the stable-order principle, the cardinal principle, the abstraction principle, and the order-irrelevance principle. A genetic decomposition summarising this organisation of counting principles is presented in Figure 3 (next page), which is then explained.

Gelman and Gallistel discussed a stage-like development of a child's counting ability; this has been described by the use of solution associations to describe the use of increasingly complex *pre-counting, simple counting* and *counting concepts to solve the problem of single collection counting*.

Pre-counting is described by the aggregation of the *one-one principle* and the *stable-order principle*. The *stable-order principle* aggregates the concept of the *numeron sequence*, itself aggregating the notion of *no tag repetition*. An inheritance association describes two more specific types of *numeron sequence:* the *conventional sequence* and the un-conventional sequence. The difference between these two numeron sequences is highlighted by an inversion association. The *conventional sequence* is described by the aggregation of *conventional numerons*, whereas the un-*conventional sequence* is described by the aggregation of *numerons*, and thus the more specific *conventional numerons* or *un-conventional numerons*. The *numeron sequence* is also a component of *synchronous tagging*, which is in turn aggregated along with the skill of *set partitioning to define the one-one principle*.

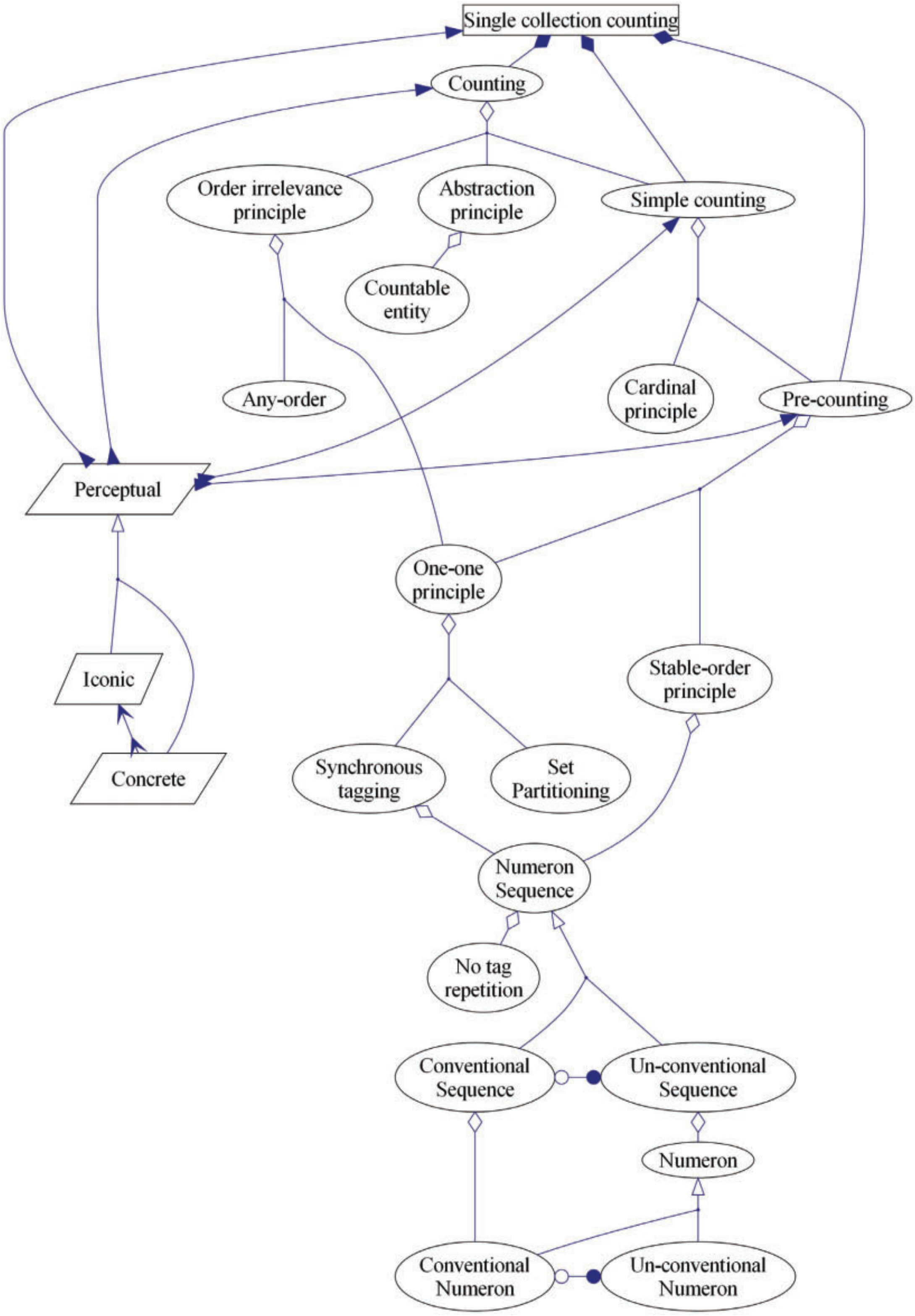

Figure 3. A genetic decomposition of Gelman and Gallistel's counting principles

The more complex concept of *simple counting* is described by the aggregation of *pre-counting* and the *cardinal principle.* The *order irrelevance* principle aggregates the concepts of the *one-one principle* and the notion of *any-order.* The *abstraction principle* aggregates the notion of *countable entity.* Together, the *order irrelevance principle and* *abstraction principle* are aggregated along with *simple counting* to describe the most complex *counting* concept.

Gelman and Gallistel's work focussed on children's counting of perceptual objects, as described by the expression association that indicates the problem of *single collection counting* may be expressed using *perceptual objects.* The perceptual objects have two more specific forms, *concrete* and *iconic,* as described by the inheritance association. Similarly, *perceptual objects* may be used to express *pre-counting, simple counting* and *counting.*

Discussion

Using the proposed alternative theoretical framework, operational model and graphical language, other early-number literature has also been analysed and described (Nutchey, 2011), including early-number word problem classification (Carpenter & Moser, 1983; Fuson, 1992), the development of number-word and number-sequence meaning (Fuson, 1992; Olive, 2001; Steffe & Cobb, 1988) and the strategies used to solve early-number word problems (Carpenter & Moser, 1983; Fuson, 1992). The resultant genetic decompositions were then synthesised together to form a composite description of early-number; a process that revealed similarities, differences and sometimes discrepancies in the literature. The resulting complex genetic decomposition, presented in Nutchey (2011), includes 56 problem objects, 49 concept objects, three representation objects, and over 200 associations to organise these objects. This activity of analysis and description has demonstrated the viability of the proposed graphical language as a tool with which to characterise World 3 mathematical knowledge. In the future, further analysis and description activity should extend the composite description to include the various representations commonly used in early-number, in particular those that scaffold the development of counting, addition and subtraction strategies.

The composite description of early-number may provide a basis for the analysis and description of an individual learner's World 2 understanding. A mechanism has been proposed (Nutchey, 2011) which suggests that World 2 understanding can be described (and thus analysed) in terms of a chronological sequence of *images* – collections of problem, concept and representation objects that each describe an activity in the learner's conceptual development. Guided by the notion of reflective abstraction, the analysis of an image sequence may lead to the assessment of a learner's understanding and the identification of future activities that may enhance their understanding.

The alternative theoretical framework, operational model and graphical language may potentially advance mathematics teaching and learning practice in several ways. The modelling technique may form the basis of computer-mediated learning environments that are responsive to learner's mathematical conceptual development. The graphical language, when used to express the highly connected nature of mathematics, may support a teacher's development of learning activities that scaffold students' constructive exploration of this organisation of mathematical ideas. This potential for theory to impact practice will be the topic of future research and development activity.

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