THE USE OF PROBLEM CATEGORISATION IN THE LEARNING OF RATIO

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Mathematics pupils in Singapore are not performing to expectation. Pupils fail to apply learnt concepts, and new concepts are learnt in isolation instead of through a 'build-up' based upon known older ones. This ongoing study investigates relating students' prior knowledge of the topic Ratio to new concepts. *Case Based Reasoning* and *Cognitively Guided Instruction* are used in this research. Their frameworks are combined, creating 'categorisation' where items are grouped, based on the concepts.

Introduction

Ratio is taught in Primary 5 and 6 in Singapore. The Curriculum Planning and Development Division (CPDD, 2007), has spelt out justifiable expectations, but these and assessment do not meet. Many pupils are unable to apply what they have been taught to solve new problems, while others are unable to reason logically or use information correctly, possibly because of the lack of effective problem solving strategies. The study reported in this paper was designed to investigate the use of categorisation and its effectiveness in solving ratio problems. Tied to categorisation is the use of solving strategies that are based on the concepts studied. In order to provide a focus for these objectives, the following research questions were formulated.

1. Does categorisation of problems result in meaningful differentiation of *student thinking* about ratio?
2. Does categorisation of problems result in meaningful differentiation of *student performance* about ratio?
3. What kinds of informal strategies do children use to solve ratio problems before and during instruction?
4. What instructional implications (teachers' and children's) can be drawn from children's pre-instructional knowledge in relation to problem categorisation?

Literature review

The Singapore Mathematics Framework (Ministry of Education, 2006) considers mathematical problem solving to be central to mathematics learning. Students are to attain and apply the mathematics concepts and skills in a wide range of situations, including non-routine, open-ended and real-world problems; however evidence has shown that problem solving is not well developed in our pupils. Kaur and Yap (1999) reported that when students were given concepts in unfamiliar situations, many did not perform as well as expected. There is a need to address the gap between students' initial ability to understand concepts and the new concepts they are to learn. To achieve this goal, the principles of *Cognitively Guided Instruction* (CGI) (Carpenter, Fennema, Franke, Levi & Empson, 1999) and *Case Based Reasoning* (CBR) (Kolodner, 1997) were adopted for the study. CGI aims for teachers to "work back from errors to find out what valid conceptions students have so that instruction can help students build on their existing knowledge" (Carpenter, Fennema & Franke, 1996, p.14). In CGI, the emphasis shifts from teachers finding ways to teach mathematical knowledge to students constructing their own knowledge based on their intuitive problem solving strategies. Supporting CGI is the idea of CBR, based on previously acquired experience.

Kolodner and Guzdial (2000) explained that the process of carrying out CBR includes: using case libraries (in this study, a collection of similar word problems) as a resource; indexing problems (identifying and classifying questions that are similar, or questions with a common concept); retrieval processes (recalling previously done questions to help solve current ones) and partial matching processes (matching similar questions to existing ones). One major issue with CBR involves the process of indexing problems. This means identifying old situations that are relevant to a new one. Suitable cases can be recalled if they are indexed well. Good indexes and the ability to apply knowledge or skills from one situation to another different situation are critical in CBR.

CGI (use of categorisation) is tied together with the CBR (instructional process flow) framework to bring about optimum learning. In the CGI framework, there are three main components: problem types, pupils' informal knowledge of strategies, and pedagogy concerns. This study focuses on only one component of CGI, namely problem types, as the ability to solve word problems depends so much on pupils' ability to recognize the differences among the problem types (Carpenter et al., 1999).

CGI therefore involves examining the various structures of problems. In ratio, problems are placed in various categories based on the distinctive feature each structure offers. Each category influences the strategies that pupils use to solve the problem. Hence, these categories are tagged not only based on their structure of questions but also on the concepts used to solve problems. There are seven categories in all. In order of increasing difficulty they are:

- (Category 1): *Ratio with Values assigned*. Example: The sides of a triangle are in the ratio of 2:3:4. The longest side is 68 cm. Find the perimeter of the triangle.
- (Category 2): *Ratio with one quantity remaining the same*: Example: The ratio of the number of Lynette's stamps to Joel's was 5:4 at first. After Joel collected another 58 stamps, the ratio became 15:14. How many stamps did Lynette have?
- (Category 3): *Ratio with a constant difference*: The ratio of Jessie's age to her father's age is 3:7. 12 years later, the ratio will be 3:5. How old is Jessie now?
- (Category 4): *Ratio with a fixed total.* Example: Ron and Kat shared some mangoes in the ratio 3:8. When Kat gave 48 mangoes to Ron, the new ratio of Ron's mangoes to Kat's was 9:2. How many mangoes did Roland have at first?
- (Category 5): *Fractional Parts of a ratio.* Example: Mark and Sean shared some marbles in the ratio 5:4. After Mark gave half of his marbles to Sean, Sean had 96 more marbles than Mark. How many marbles did they have altogether?

- (Category 6): *Ratio with changing quantities.* Example: The ratio of the number of men's watches to the number of ladies watches in a showcase was 4:1. After putting another 48 men's watches and 36 ladies watches into the showcase, the ratio became 8:4. How many watches were there at first?
- (Category 7): *Ratio of a ratio.* Example: At a party, the ratio of the number of boys to the number of girls is 3:2. If each boy and each girl is given stickers in the ratio 2:3, a total of 1992 stickers are needed. How many boys and girls are there?

Recognizing the differences among the problems alone is insufficient – pupils must be able to apply the correct strategy to solve the problem. Applying the correct strategy comes about from being able to identify the concept within the problem. To apply the correct strategy, pupils must first overcome conceptual difficulty. Lo and Watanabe (1997) claimed that technical difficulties usually are not the main obstacle in curtailing students' solving process. Conceptual difficulty apparently is much greater and more complicated. Categorisation can solve this problem of conceptual difficulty, as it trains pupils to identify concepts involved in a particular question. Lamon (1993) believed that there is a need to move beyond identifying the litany of tasks variables that affect problem difficulty, toward the identification of components that offer more explanatory power for children's performance. That is, there is a need to do more than just look into pupils' cognition. Combining frameworks of CGI and CBR could eliminate a litany of tasks variables.

Research method

Sample: Thirty-two Primary 6 pupils from a primary school in Singapore were selected to form the non-random purposive sampling group. A teacher also participated. The school was co-educational, with the ratio of boys to girls being approximately 1:1. Most of the students' mathematics ability met the nation's national average. None of the pupils had been exposed to problem categorisation. In Primary 5, these pupils used various heuristics such as model drawing, listing, and guess and check to solve ratio problems—part of the heuristics package used by the school. Categorisation was a newly developed process and had not been tried with these pupils before.

Research Design: Mixed methods were adopted where quantitative and qualitative data were simultaneously collected and merged. Creswell (2008) believes that the strengths of one data form offset the weaknesses of the other form. Both forms of data were collected at the same time and the results were used to validate each other.

The research was carried out in three phases:

Phase 1 (Categories 1, 2 and 3): A pre test was administered before the start of phase 1. This phase dealt with pupils' first interaction with ratio and only basic ratio categories were covered. New concepts were built on the old ones using the CBR lesson flow framework. In this phase, the effectiveness of categorisation for the basic ratio problems was being investigated. Strategies used were being considered in order to determine if pupils were able to move away from model drawing to using ratio concepts. Difficulties and misconceptions pupils faced were also examined. To do this, the use of voice recording with MP3 players was adopted. Pupils recorded their thought processes into the device and the recording was played for the whole class to listen to. This is where students' peers, the teacher, and researchers heard their thought processes while solving ratio problems. Discussion was opened for all to comment constructively on what they heard. Misconceptions that arose at that point of time were rectified, so pupils were made aware of the correct concepts and approaches to solving ratio problems.

Phase 2 (Categories 4 and 5): This intermediate phase was one where pupils had become familiar with the lesson structure. The teacher used questioning more in this phase in order to probe thinking at a deep level and to delve beneath the surface of ideas.

Phase 3 (Categories 6 and 7): Strategies applied (from simple model drawings/guess & check/listing→ Ratio concepts→ algebra) were expected to be more sophisticated as students moved away from their initial (informal) strategy. A post test (7 questions) was administered in this phase. MP3 recording, written formative tests (4 questions each), interviews (with the teacher and 4 – 5 pupils) and journal writing were carried out in all three phases.

Data collection

(i) Classroom observations: Classroom observations were conducted, where the researcher/observer blended into the setting, 'becoming a more or less 'natural' part of the scene' (Bogdan & Biklen, 2003). Pre and post lesson discussions were carried out to probe the teacher's personal opinions and pupils' conceptual understanding of each lesson.

(ii) Student interviews: Interviews generally allow for open-ended responses and are 'flexible enough for the observer to note and collect data on unexpected dimensions of the topic' (Bogdan & Biklen, 2003). Pupils were interviewed to bring out their knowledge of ratio. All interviews were tape recorded and transcribed.

(iii) Journal Writing: This process, as mentioned by Yazilah & Fan (2002), is a good avenue for pupils to provide feedback on mathematics teaching and learning (Fan, 2006). In this form of assessment, questions were asked in written form to determine factors such as students' feelings, difficulties, discoveries, and thoughts.

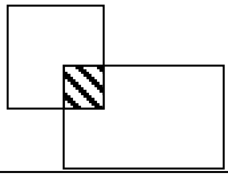
(iv) Performance Test: A written test measured pupils' explicit understanding and performance of ratio concepts through problem categorisation. Making them solve problems in a pen and paper test served as reaffirmation of their understanding. This way, there was a basis for determining whether an individual's ability had changed (Malone, Douglas, Kissane & Mortlock, 2007) and whether problem categorisation was effective in developing ratio concepts. Two forms of written tests were given: formative and summative (in the form of a post test). A scoring scale on marking and measuring problem solving (Malone et al., 2007) was used.

Findings

Changes in student thinking

Voice recording through MP3 player: When transcripts of pupils' thought processes during pre and post test were compared, it was found that there was a noticeable change in student thinking. In the post test, pupils were able to reason logically and correctly, hence they were able to categorise questions correctly. By doing so, pupils get to the concepts tied to the category identified. One example follows (Figure 1).

The figure is made up of a rectangle and a square. The ratio of the area of the square to the area of the rectangle is 1 : 3. A shaded area of 20cm² is being cut out. The ratio of the area of the unshaded square to that of the unshaded rectangle is now 2 : 7. What is the length of each side of the square?



Pre-Test

(7 – 3) parts → 20cm²

1 part → 20cm² ÷ 4

= 5cm²

7 parts → 5cm² × 7

= 35cm²

35cm² + 20cm² = 55cm²

55cm² ÷ 2 = 27.5cm

Note: Pupil had a misconception in area and this misconception was not corrected because the focus of this study was on ratio.

Post-Test

[Constant difference because equal parts are taken away]

Before	**After**
S : R	S : R
1 : 3	2 : 7
5 : 15	4 : 14

(5 – 4) parts → 20cm²

1 part → 20cm²

4 parts → 20cm² × 4 = 80cm²

80cm² + 20cm² = 100cm²

100cm² ÷ 2 = 50cm

Figure 1: Pre- and post-test solutions to question 3

Pupil C12 : Transcript for Pre-Test question 3

> 7 – 3 because they are the two parts for rectangle. One is before cutting and the other after. So I minus them to get the shaded area that has been cut. Then the answer is equal to the shaded area which is 20cm². I find 1 part first and then multiply it by 7 parts to get the unshaded area. Answer is obtained by adding the cut part to the unshaded part. This gives me the area. Then I divide by 2 to get each side.

Pupil C12 : Transcript for Post-Test question 3

> The same parts are taken away. So, this is constant difference. I must make the difference between the before and after ratio to be the same. Then I can see that the old part for rectangle is 15 and the new part is 14. So the difference is 1. The old part for square is 5 and the new part is 4. The difference is also 1. Now, the difference is the same. So, 1 part is 20cm². Then I find the unshaded part of square, 4 parts and multiply it by 20 cm². This gives me 80 cm². I must add 20cm² to 80 cm² to give the area of a whole square. To find the length, I divide the area by 2.

Based on the thought processes revealed, all students (32/32) managed to reason correctly, placing problems in the correct categories 1 to 3 and obtaining correct solutions. The reasons used to identify the three categories were also correct.

In category 4, 93.75% of the pupils (30/32) managed to identify the category for the given problem. One pupil said that he knew the concept after reading the question, but found the headings to the categories difficult to recall.

In category 5, pupils' reasoning was very good, with almost everyone doing well in this category. In the MP3 recording every pupil mentioned the use of 'lowest common multiple'. One pupil (C6) actually made a comment that category 5 (fractional parts of ratio) is usually tied to another category. The following describes his experience.

> Mark and Sean shared some marbles in the ratio 5:4. After Mark gave half of his marbles to Sean, Sean had 96 more marbles than Mark. How many marbles did they have altogether?

Pupil C6 : Transcript for Pre-Test question 5

> I get half of 5 and add it to 4 to get 6½. This leaves Mark with…2½. Then I minus 6½ with 2½ to get 4 parts. This 4 parts is the "more" parts which is the same as 96. Then I find 1 part and multiply it by 9.

Pre Test			**Post Test**		
Before	**After**		**Before**	**After**	
M : S	M : S	4 parts $\rightarrow 96$	M : S	M : S	(13 – 5) parts $\rightarrow 96$
		1 part $\rightarrow 96 \div 4 = 24$	5 : 4	? : ?	1 part $\rightarrow 96 \div 8 = 12$
5 : 4	$2\frac{1}{2} : 6\frac{1}{2}$	(5 + 4) parts $\rightarrow 24 \times 9$	10 : 8	5 : 13	(5 + 13) parts $\rightarrow 12 \times 18$
		= <u>216</u>			= <u>216</u>

Figure 2: Pre- and post-test solutions to question 5

Pupil C6 : Transcript for Post-Test question 5

> I must find a common multiple of 2 and 5 first. It is 10. Then I must multiply 2 to 5 to get 10 parts. I must also multiply 4 by 2 to get 8 parts. So, the new ratio is 10:8. Now, half of 10 is 5 parts. This 5 parts must be given to Sean. Now, he has (5 + 8) parts ….

The pupil worked quietly and started to complete the solution. Later during an interview, both solutions (pre and post tests) were put to him (Figure 2) and he was asked which of the two he preferred. He quickly pointed to the post test and said that he did not like to work with fractions. He added that sometimes, when he had to divide the value (points to 96) by a fraction, he usually 'messes up' his answer by getting it wrong. With this new approach of categorisation, he felt that he did not have to work with fractions at all. Also, he commented that this problem can also be a Cat 4 (total before = total after) problem. He noticed that category 5, fractional parts of ratio, usually comes accompanied by another category.

In category 6, 75% (24/32) of the pupils were able to identify the correct category and 62.5% (20/32) managed to obtain the correct solution. One particular reason for this was because the category involves the use of algebra. Those who managed to get this completely correct used basic algebra to solve this. The rest did it by algebra too, but were stuck halfway through the working. When MP3 recordings of 12 pupils who failed to get this question correct were played, it was discovered that all of them could identify the correct category and the concept, but were stuck when it came to the technical part of algebra where they could not manage when they transposed to the other side of the equation. They worked through the equation using their understanding of equivalence and constructed their knowledge based on their intuitive problem solving strategies.

In the last category, pupils were able to identify this category as the only one with two given ratios that do not refer to the same thing (Figure 3, example 1). In the post test, pupils were able to get through this problem easily. All of them managed to get this correct as they were quick to notice two given ratios that represent different items; pupils and stickers. However, in the formative test that was given at the end of the third phase, pupils were not able to solve a particular question (Figure 3, example 2).

In the MP3 recording, most of them could not find the other ratio; G : B : T = 10 : 5 : 100. It did not cross their minds that the amount of money donated can be written as a ratio. Because of this, the class was not able to obtain a solution. More scaffolding was done to help address the gap between students' initial ability to understand concepts and the new concepts they had to learn in category 7. From then on, pupils were more aware (as found in the transcripts) of the 'other ratio' in category 7.

Example 1	Example 2
At a party, the ratio of the number of boys to the number of girls is 3 :2. If each boy and each girl is given stickers in the ratio 2 : 3, a total of 1992 stickers are needed. How many boys and girls are there?	The ratio of the number of girls to the number of boys to the number of teachers in a school is 5 : 6 : 1. Each girl donates $10, each boy donates $5 and each teacher donates $100 in a fund-raising event. If a total of $27000 is donated, how many pupils are there in the school?

Figure 3: Examples in Category 7

Interviews: Five pupils who were interviewed said that learning where new concepts were built on old ones made learning ratio easy. These pupils were able to link new knowledge to the old. This is important as proportional reasoning becomes more complex and detailed as pupils go deeper into the categories of varying content. To overcome this, each category must scaffold the next. This way, pupils are able to see that all ratio problems are connected, and that concepts build from prior knowledge instead of new ideas, thus encouraging a transfer of knowledge between categories.

It was noticeable that pupils began to apply analogical reasoning that focused on reasoning based on previous experience in category 3. They were able to explain, correct and engage the teacher during lessons and had cultivated the habit of reading a question seriously as they realised that each question contained clues to the answer. They were now more conscious than before of the importance of reading to understand. When asked if they faced any difficulties, two felt that the headings of each category were difficult for them to recall. Another two felt that category 5 was very difficult, but were quick to say that if they could overcome that, they would be able to solve more ratio questions. All of them agreed that categorisation should be practiced in other topics, especially in fractions.

Journal writing: In this qualitative aspect of the study the three items discussed were: (i) student thinking, then and now, (ii) preference; categorisation versus current heuristics, (iii) confidence in solving ratio questions.

(i) Student thinking, then and now: Pupils were asked if there were changes in the way they thought when solving ratio questions before and after categorisation. Everyone agreed that there was a change after the categorisation intervention. Unlike before, when they attempted ratio questions after the intervention, they first read before deciding on the category. They realised that they were indirectly 'forced' to read the question. Also, the availability of a strategy (that comes from the concept) was a plus for them. This way, time spent deciding on a strategy was saved and put to better use. Six pupils (18.75%) commented that before, the given information meant very little to them and they did not know what to do with it. The reason for this was that the pupils did not understand the question, hence the information was not fully and correctly utilised. Now, with categorisation, pupils were starting to read the question for understanding first. Twenty-eight pupils (87.5%) agreed that categorisation also helped them give structure to their thinking and made the solving process easier.

(ii) Preference – categorisation versus current heuristics: In the journal entries collected, everyone agreed that categorisation helped them solve ratio questions better than the usual heuristics. The features each category contained [e.g. receiving the same amount → constant difference (cat 3), A shares with B → total before = total after (cat 4)] made it easy for them to identify questions and match them to the respective categories. Each category had a concept that led to a solution. The pupils felt that knowing *what* strategies to use and *how* to use them gave them the confidence they needed to solve the questions. This addressed the issue of *how* learning was facilitated.

(iii) Confidence: It was found that all pupils were more confident in solving ratio questions. With categorisation, pupils did not have to worry about finding the right strategy as this was tied to the categories. Pupils became more confident as they had to only focus on getting to the correct category. Pupils were able to communicate their solutions clearly and logically. It was noticeable that the high performers' explanations were very short and to the point, mentioning only the relevant points, whereas the average performers were very detailed and systematic in their explanation. The low ability students struggled to explain their steps and they took longer to record their thoughts.

Changes in student performance

On the whole, pupils' performance in the post test improved, especially in categories 6 and 7 (Table 1).

Table 1: Student performance on pre and post-test

	Pre-test		**Post-test**	
Categories	**Number of pupils**	**%**	**Number of pupils**	**%**
Category 1	32	100	32	100
Category 2	25	78	32	100
Category 3	19	59	32	100
Category 4	13	41	30	94
Category 5	15	47	30	94
Category 6	0	0	20	63
Category 7	0	0	32	100

Strategies used

Most pupils started ratio using model drawing; something they have been taught to do since primary 1. Therefore, it was not surprising that most pupils used model drawing in the pre test. Those who were neither good nor confident in model drawing used guess and check and listing as alternatives. These pupils usually fall in the group of low performers in mathematics. As opposed to the post test none of the 32 pupils solved any of the pre-test questions using ratio. Surprisingly, no model drawings were used in the post-test. Listing, and guess and check were heavily used by the low performers in categories 4, 6 and 7. These pupils managed to apply the ratio concepts learnt to solve categories 1, 2, 3 and 5. It was noticeable that these pupils reverted to their old strategies when they were faced with either an unfamiliar or difficult question. A big shift in strategy was seen in the average performers. These pupils were able to move away from direct modelling to writing it out. They found category 6 very challenging as they struggled with the use of algebra. The use of negative numbers and the transformation of the operations were the reasons why they were unable to solve category 6 questions, but the concept in that category was fully understood as shown in the post test.

Implications

Pupils gave positive feedback on learning ratio through categorisation. The results of the post-test confirmed the value of this approach. Through the journal entries, it was clear that pupils hoped to have more learning conducted this way.

From a pedagogical point of view, the teacher participant felt that it was easier to teach ratio this way and thought that it would be good to extend this approach to other topics in mathematics. One particular aspect she liked was the construction of new knowledge from old. She felt that this form of learning was effective and it formed a strong foundation in pupils' learning of ratio. She also noticed that her pupils were beginning to read the questions, something she had been asking the pupils to do with little success. In addition, she found the environment in her class more active as, unlike before, pupils were participating in the discussions. She also noticed a positive change in pupils' reasoning skills.

Resources used were carefully selected and studied before the categories were decided on. Obtaining questions was easy, but identifying the concepts in the questions was not. Concepts identified had to be vetted to ensure that they could be applied to all questions in that category. In short, considerable effort went into planning the resources in order to achieve the desired outcome.

Pupils do not have to worry about coping with new information regardless of their readiness when using the approach described in this paper. CBR relies heavily on prior knowledge and appears to work well with CGI problem categorisation and shows promise for assisting students towards a better understanding of learning the mathematics of ratio.

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