LANGUAGE-RELATED MISCONCEPTIONS IN THE STUDY OF LIMITS

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This paper reports on the language-related misconceptions of a group of post-secondary students when working on problems involving the limit of a real-valued function at a single point. In this qualitative study, 50 post-secondary students took a test and participated in a survey, from which 10 were interviewed after the test. The data revealed several misconceptions held by the post-secondary students about the limit concept that were related to the issue of language. Such language-related misconceptions resulted from incorrect *internal representations* and the *inability to reify* the limit.

Analysis is the most important area in mathematics, where students have to learn concepts that are linked to the notion of limit of a function at a point. From the understanding of the limit concept, other fundamental concepts like continuity, differentiability and integrability are all established. Hence, the limit concept underscores almost every branch of mathematical analysis and can be studied in various settings. As Huillet (2005) stated:

> The limit concept can be studied in many different settings: geometrical (area and volumes), numerical (sequences, decimals and real numbers, series), cinematic (instantaneous velocity and acceleration), functional (maximum and minimum problems), graphical (tangent line, asymptotes, sketching the graph of a function), formal ($\varepsilon - \delta$ definition), topological (topological definition, concept of neighbourhood), linguistic (link between natural and symbolic languages of limits), algebraic (limits calculations). Each of these settings underscores a specific feature of the limit concept. (p. 172)

Historically, the idea of limits resonated since the Greek era, around 600 BC. The Greeks however focused on results and the idea of limits was used only intuitively. In the 17th Century calculus, the notion of limits came to the fore through the works of Newton and Leibniz. However, the idea of calculus rested on weak foundations. It was only about 150 years later that the rigorous definition of the limit was constructed through the works of Cauchy and Weierstrass.

The term *limit* used in this paper means the limit of a function at a point, unless otherwise stated. In other words, $\lim_{x \to a} f(x) = L$ means, for every ($\forall$) $\varepsilon > 0$, there exists ($\exists$) a $\delta > 0$ so that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$ and $x \in$ domain of f. It is to be noted that the limit of a function is defined to exist, if the left hand and right hand limits both exist and are equal. As can be seen from the definition above, grasping the idea of limit requires students to decode its meaning from a relatively complex symbolic statement.

Educators and students face the hard transition that is necessary to leap from the routine to the non-routine aspects of mathematics where limits are first encountered. This transition is hard because the limit concept represents a concept that requires advanced mathematical thinking processes (see Dreyfus, 1991, pp. 35–36). Such advanced thinking processes calls for the assimilation of three key characteristics. These are, *generalization* (to derive from particulars), *synthesizing* (process of merging into a single picture), and *abstracting* (transition from the concrete to the abstract). Students require new methods to assist them in making the transition from the secondary to post-secondary level mathematics.

Given the complex nature of the definition of the limit of a function at a point, it is not surprising that some students develop misconceptions about limits. Thus, it is fundamental for educators to investigate what the misconceptions are and why the misconceptions occur. A study on the misconceptions arising from limits may provide reflection into how curriculum should be designed and how teaching of limits should be carried out.

This paper will address two research questions.

1. What kinds of language-related misconceptions are there when students study limits?
2. Why do such misconceptions occur?

The term *misconception* as used in this study refers to the reason which constitutes the basis over which an error is made, with reference to the individual student's perspective. "The misconception which forms the basis of the observed error may lie in the child's conceptual knowledge or knowledge store or in the strategies which are developed in order to handle the problems under study" (Booth, 1983, p. 32). Another definition of misconception as noted by Ferrini-Mundy and Lauten (1993), describes misconceptions as *non-traditional student views* (see p. 156).

Literature review

The language issue in the study of limits has been investigated previously. Monaghan (1991) stated that misconceptions in learning limits arise as a result of the language used in limit terminology. He reported ambiguities that arose primarily from four phrases, namely: *approaches, tends to, converges*, and *limits.* These terms were cause for ambiguity because students formed their own interpretations from the four terms mentioned. Students used speed limit to rephrase limit when asked to replace the word limit within another context. The word *limit* was seen as a boundary which could not be exceeded. The word *converges* was construed in the context of lines converging to lines, but not to numbers. The terms *approaches* and *tends to* seemed to give the students the impression that the limit is a dynamic concept. Accordingly, it can be said that all these language misconceptions stemmed from the everyday meanings of the word being used in limits. Monaghan added that: "Students should be led to explore and discuss their conceptions and to realize how everyday meanings of mathematical phrases can direct them into fallacious interpretation" (p. 24).

Monaghan (1991) also studied the consequences the use of language has on teaching mathematics. When English phrases like *tends to* and *approaches* are used in mathematics, the terms have a different meaning in mathematics. The mathematical language is a precise one in contrast to the spoken English language, and in addition, students seem to attach their own meanings to things. In fact, according to Quine (1968) understanding things involves interpreting meanings. Such interpretation also entails translation. Quine argued that this translation is relative to each individual and thus indeterminate. Hence, students who attach meanings to concepts may simply be interpreting what they translate the meaning to be. Consequently, meanings are subject to the students' ontological relativity.

Davis and Vinner (1986) argued that misconceptions will continue to proliferate as long as the word *limit* is used too early in the calculus syllabus. They proposed that using the word *associated number* - a neutral phrase in place of limit, at least at the onset of a calculus course might help. For example, for the sequence $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ instead of asking for the limit of the sequence, one could ask for its *associated number*. Swenton (2006) argued that many difficulties that occur in the study of limits are caused by inadequate mathematical language: "we argue that a large number of the difficulties, both specific and general, that occur in the instruction of limits stem from the lack of a mathematical *language* that properly addresses the fundamental nature of limits conceptually, computationally and logically" (p. 643). Swenton proposed using *near-numbers* as a language for limits (see p. 644 for details).

On the other hand, Schwarzenberger and Tall (1978) added that the technical language used is a matter that may create conflicts in learning. Schwarzenberger and Tall divided conflicts into conscious and subconscious ones. These authors also queried that when we say 'make the *n*th partial sums as close to *s* (its limit) as we please, by making *n* sufficiently large', what precisely is the meaning of that? Ambiguity arises as to 'how large is large' and 'how close is close'. A cognitive subconscious conflict arises here as the term *close* means near but not coincident. Hence, a misinterpretation may occur; namely, the *n*th partial sums can be very near to *s*, but never equal to it. Skemp (1986) added that, "we should never use convenient but loose phrases such as 'as small as we like' " (p. 67).

Epp (1999) contributed to the issue of language in mathematics, by bringing in the aspect of quantification. Some students, she argued, get confused between the everyday usage of certain words and their technical meanings in mathematics. Students are not able to spontaneously read into the truth or falsity of universal and existential statements. Evidence showed that the conditional statement 'If A, then B' is misconstrued as being equivalent to the converse; 'If B, then A'. Epp highlighted that textbooks have a part to play in the formation of this errors. Williams and Irving (2002) discussed about the discourse people subscribe to at different times. They contended that different language registers allow for the construction of different universes of meaning (see p. 209). Hence, English and mathematics domains are mutually exclusive when it comes to certain terms involving limits where meanings have to be defined precisely.

The disadvantages of relating mathematical language to the English spoken language have been highlighted. However, the aforementioned relation can be advantageous as well, when first presenting the limit concept. Gass (2006) used the word *approaches* on purpose instead of *converges* to establish a familiarity between the mathematical and English languages. “I prefer ‘approaches’ rather than ‘converges’, because it maintains a familiar-setting language while we take on the challenge of limit definitions and limit proofs” (p. 148).

Methodology

This paper reports on data collected from a larger qualitative study involving 50 post-secondary students about the students’ misconceptions in the study of limits. These students had taken an introductory course in calculus that was taught during their first year in a private university in Singapore. A test comprising of 3 items each with several parts on the limits of a function at a single point was administered to the students. The functions included covered continuous, piece-wise continuous and discontinuous functions. The functions were also represented differently; for example, some were represented by formula while others by tables or graphs. Rational, modulus and floor functions were among the functions included on the test.

The data responses were collated and coded according to the type of error made. Subsequently an interview schedule (comprising of 4 items) was designed based on the errors observed on the test and 10 of the students were interviewed. The 10 students interviewed were chosen based on the different kinds of errors they made. In particular, only the students who made language-related errors will be the focus in this paper. The emphasis accordingly lies on the reasons why students faced language difficulties.

A survey was also carried out based on the interview data to probe further into the validity of some of the language misconceptions.

Results and discussion

The data collected from the participants included: test scores, test scripts, a survey and interviews. In discussing the two research questions that follow, this paper will focus primarily on the results from the test scripts and interview data. Due to space consraints, only a few cases are highlighted here.

What kinds of language misconceptions are there when students study limits?

The language error appeared on a number of items of the main test. In particular, some of the respondents used terms such as *indeterminate, non-applicable, indefinite, undefined* and *does not exist*. In this study, some responses included regarding $\frac{0}{0}$ as a limit that *does not exist, is undefined, is indeterminate,* or *is non-applicable*. In addition, some students also wrote *indefined*. The response *indeterminate* was partially correct, however it had to be simplified into a limit that could be determined; often, this simplification was not performed and *indeterminate* was left as the final answer. Considering some of the students in the interview sample, Brooke and Ivy are low-ability students while Joseph a middle-ability one. Brooke during the interview described the limit as infinite. “Ah, infinite, does not exist at all”. Ivy thought that infinity was similar to indeterminate. Joseph on the other hand regarded the term *does not exist* to be similar to *indeterminate* and *undefined*. It is evident that the students in this study possess varying interpretations of the words *infinite, does not exist, indeterminate, indefined,* and *non-applicable*. In the literature, Monaghan (1991) uncovered that the terms: *approaches, tends to, limit,* and *converges* were all taken to mean different things because of the physical connotation each term entailed. For example, limit meant a boundary that could not be exceeded. Thus, the language issue is a factor that led to incorrect limit values. The students are not aware of when to use the words *infinite, does not exist,* and *indeterminate*. Brooke for example thought that *infinite* meant the same thing as *does not exist*. Responses from some students showed that the term *approaches* was misconstrued as synonymous in meaning to *approaching from the left hand side*. The following interview extract shows that some students assumed that it was acceptable to attach their own meaning to certain terms (in this case approaches means approaches from the left). During the interview, a particular student Mary was asked for the limit as x approaches 5 for a function defined by: $f(x)=1$ for $x>5$ and $f(x)=-1$ for $x<5$. Mary responded as follows.

I:	What about for this graph, the limit as x goes to 5?
Mary:	As x goes to 5, it will, as x goes to 5, negative 1.
I:	Why negative 1?
Mary:	Because I view it in such a way that approaches means this way you see.
I:	Meaning comes from the left?
Mary:	Ah, normally when you say its approaching something means it's from a smaller number.

It is evident from the response of Mary above that the notion of 'smaller to bigger' factors into the meaning of the term *approaches*. Hence entailing inherently the concept of ordinal numbers which is then associated with the limit.

Why do such misconceptions occur?

The language of limits needs to be clarified. It is possible that confusion is caused because, in real numbers sometimes people write $\frac{a}{0}=$undefined and sometimes $\frac{a}{0}=\infty$ for $a\neq 0$.When this real number theoretical fact is imported into the limits domain, the misconception that '$\infty=$undefined' is formed. Regarding limits, '∞' represents unboundedness and *does not exist* means that the left and right hand limits are different. David, a high-ability student responded on item 1(v) of the main test (see Figure 1) that the limit was 0 and this was similar to saying that the limit does not exist. David's statement clearly shows that, the notion of a limit that does not exist is not well understood.

Item 1(v): Find: $\lim_{x\to 0}\lfloor x\rfloor$ v) 0 ∴ it does not exist

Figure 1. Response from David on item 1(v).

Schwarzenberger and Tall (1978) argued that technical language used can create conflicts in learning and some of these conflicts are subconscious ones. Hence, this sample demonstrated learning conflicts that stem from language in real numbers being regarded as similar to language in limits. While Monaghan (1991) showed that terms

like *tends to* and *converges* are taken to have different meanings, the data in this study show that the value zero is assumed to be synonymous to 'the limit does not exist'.
On the other hand, Goldin (1998) claimed that certain words may bring out images in the minds of the students. "For instance, words and phrases not only have grammatical and syntactic structure; they evoke non-verbal images" (p. 144). Some students in this study have linked certain words to represent certain objects; such as linking the phrase *undefined* to the indeterminate form $\frac{0}{0}$. The phrase undefined has possibly evoked the image of representations of fractions where the denominator is zero.

The difficulty encountered with terms such as *indeterminate*, *non-applicable*, *indefinite*, *undefined,* and *does not exist* can also be analysed through a learning metaphor highlighted by Sfard (1998) which is explained as follows. Sfard claimed that the Acquisition Metaphor (AM) involves acquiring knowledge based solely from the individual standpoint. The focus of AM with respect to learning has the individual as the emphasis. Accordingly, the respective students could have acquired knowledge of the terms such as *indeterminate*, by acquiring the knowledge on their own. The misconception that *indeterminate* has the same meaning as *undefined*, is a misconception that might have been constructed by the students themselves. The acquisition of knowledge as analysed through the AM, takes place only between the student and the terms. There is no facilitation to correct or check student understanding. Hence in the absence of teacher intervention, it is likely that language-related misconceptions may manifest.

On the other hand, the students who did not have language-related misconceptions with terms such as *indeterminate,* could have learnt or acquired knowledge of terms by reaffirming their acquisition with some external authority. The acquisition where learning is checked can be analysed through another learning metaphor put forward by Sfard (1998); namely, the Participation Metaphor (PM). The PM explains that knowledge acquisition occurs through participation with the mathematical community. Students who had their learning facilitated probably did so using textbooks or inquiring with their teachers. Such participatory learning could account for students who were able to distinguish the differences between the terms *indeterminate* and so on. Essentially, two different learning approaches yielded two different learning outcomes. The AM: where learning takes place individually versus the PM: where learning involves student participation and interaction with others.

Moru (2009) argued that responses from her interviews with 15 first-year undergraduate mathematics students included thinking of $\frac{0}{0}$ as 1, 0 or ∞. The responses were accompanied by statements like 'it's undefined' or 'the limit does not exist' (see p. 441). A particular student of Moru's, S126, claimed that $\frac{0}{0}$ does not exist and $\frac{0}{0}$ is ∞ because anything divided by 0 is ∞ (see p. 441). S126 thus thought that '∞' meant the same thing as 'does not exist'. The language factor persists in the response by S126 since he or she attached similar meanings between ∞ and 'does not exist'. Moru claimed that *generalization* is the epistemological obstacle accounting for the misconception. Accordingly, it is possible that the students in the present study have made generalizations of their own when they surmised that classifying '$\frac{0}{0}$' as *undefined* or as *does not exist* means the same thing.

The data revealed that some students thought the limit value was an approximation. For example, responses showed that the limit was a range or an estimate. Using Dubinsky and McDonald's (2001) theory (see Dubinsky & McDonald, 2001, p. 277) of Action, Process, Object, and Schema (APOS), it can be argued that students who were able to find the limit value successfully made the transition from the Process stage to the Object stage. Successful transition refers to the sequence of approximations (the Process) evolving into finally the limit value (the Object). In other words, successful transition calls for *reification* to occur. The students who left the limit as an approximation regarded the Process and Object stages as similar. Thus, students who were able to go one step further to the Object stage were successful in the computation of the limit. Those students who left their answers as an approximation could have done so because of language factors. For example, terms like 'tends to' or 'approaches' as Monaghan (1991) argued were regarded as having different meanings. The limit in this sample is seen to be an approximation possibly because of words such as 'approaches', which carries the connotation of never being reached. Hence, approximations seem reasonable if students subscribe to the English meaning of 'approaches' in contrast to the mathematical meaning. Consequently if language precedes mathematics, then the limit is not *reified*.

Conclusion

The key reasons for the language misconceptions are a *lack of proficiency in the English language* and the inability to *reify* the limit as an object may have contributed to misconceptions. Other factors accounting for the misconceptions include *strong internalized behaviour* and *knowledge acquisition* through individual and non-participatory modes of learning. Reasons such as *generalization* (a particular epistemological obstacle) and following *normal behaviour* in computing limits subject to everyday meanings of the English language, may have also contributed to misconceptions. It should be noted that certain presentations (e.g., use of colloquial language) in textbooks are also responsible for the formation of misconceptions (see Kajander & Lovric, 2009, p. 175).

Accordingly, the planning of courses on limits will have to consider carefully the role of language in the study of limits. The terms highlighted above have to be explored in greater detail with students. Clear distinctions have to made about the meanings attached to terms such as *indeterminate*, *undefined*, *does not exist,* and so on. While in any research an attempt is made for the study to cover as large a scope as possible, limitations would nonetheless exist. In this investigation, the results from the data collected may not necessarily be generalized to a wider population because of the specific nature of the sample in this study. Language-related misconceptions derived from this sample may not be similar to those arising out of other samples of students with different mathematical backgrounds. Looking forward, it is therefore timely to recommend a study on language-related misconceptions on limits to be conducted on students with different samples with varying mathematical backgrounds.

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