CHALLENGING TRADITIONAL SEQUENCE OF TEACHING INTRODUCTORY CALCULUS

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Despite considerable research with students of calculus, rate and hence derivative, remain troublesome concepts to teach and learn. The demonstrated lack of conceptual understanding of introductory calculus limits its usefulness in related areas. Since rate is such a troublesome concept this study trialled reversing the usual presentation of introductory calculus to begin with area and integration, rather than rate and derivative. Two groups of first year tertiary students taking introductory calculus were selected to trial the effect of changing the sequence; a control group and a group which followed the reversed sequence. Two-sample t-tests undertaken in Minitab on the examination results indicate there is no significant difference between the examination results of the two groups. These results indicate that changing the sequence of delivery was not detrimental to the development of conceptual understanding of introductory calculus.

Background

Rate is an important mathematical concept that is often poorly understood by many people. It is a complicated concept comprising many interwoven ideas (see Figure 1) such as: change in a variable resulting from a change in a different variable; the ratio of two numeric, measurable quantities; constant and variable rate; and average and instantaneous rate. It expresses the change in the dependent variable resulting from a unit change in the independent variable, and involves the ideas of change in a quantity; co-ordination of two quantities; and the simultaneous covariation of the quantities (Thompson, 1994).

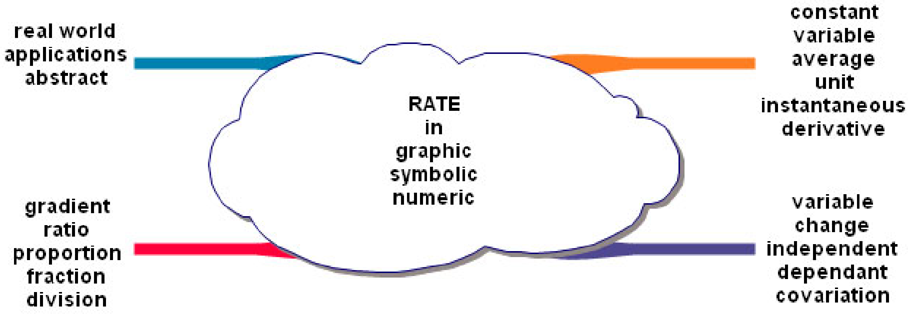


Figure 1: Complexity of the mathematical concept of rate.

Rate is strongly connected to other mathematical concepts, such as ratio; proportion; fraction; division; gradient; and derivative. In addition, rate may be seen as a purely abstract mathematical notion or embedded in the understanding of real-world applications. Rate is considered constant if the way in which the quantities change in relation to each other remains the same and variable if it differs. For example, speed may express a constant relationship between the distance and time, whilst this relationship varies when speed is changing.

Despite considerable research with students of calculus, rate remains a troublesome concept to teach and learn (see for example Orton, 1983; Ubuz, 2007). Calculus students' difficulties with rate manifest in many forms. One of the most significant of these is the confusion between the rate and the extensive quantities that constitute it (Rowland & Jovanoski, 2004; Thompson, 1994), for example understanding of speed as a distance. Thompson, reporting on the understanding of rate of his study of nineteen advanced mathematics tertiary students, describes students' confusion between the notions of change and rate of change and also confusion between amount and change in amount, for example when discussing the manner in which the volume of an inverted cone changed with height. Rowland and Jovanoski's study of the understanding of rate of fifty-nine first year science students with previous experience with calculus, found that many students confused amount and rate, for example they report one student's response that the constant term in a differential equation "represents the amount of drug going into the patient's body" or the constant term in a differential equation "is the initial amount of drug in the body" (p. 511). They suggest this confusion often resulted from a reliance on constant rate ideas not valid in the differential equations presented. Their findings indicate that the confusion between amount and rate, noted by Thompson (1994) still persists.

Other difficulties with understanding of rate include: confusion relating to symbols and their use as variables (White & Mitchelmore, 1996); lacking awareness of the relationship between slope, rate and the first derivative (Porzio, 1997); misunderstandings related to average and instantaneous rate (Hassan & Mitchelmore, 2006); related-rates problems in speed (Billings & Kladerman, 2000); and geometric contexts (Martin, 2000). White and Mitchelmore (1996) warn that symbolic manipulation may limit students' understanding to algebraic symbols and routine procedures. Hassan and Mitchelmore (2006) report on their study of fourteen Australian senior secondary students' understanding of average and instantaneous rate, suggesting that the students' previous introduction to calculus had not influenced the students' understanding of average rate or instantaneous rate. They emphasise the importance of a sound understanding of average and instantaneous rate before teaching more advanced concepts. Bezuidenhout's (1998) study involved five hundred and twenty-three South African first year calculus students and investigated these students' understanding of rate. He claims that the main confusion about rate involves the "relations between the concepts 'average rate of change', 'average value of a continuous function' and 'arithmetic mean'"(p. 397). Similarly, Oliveros (1999) states that rate was often seen as a numerical operation, similar to the treatment of rate in early secondary years, rather than a relationship between quantities.

Since the 1980s concern has been expressed regarding the difficulties some calculus students have with the concept of rate. The older research is cited to emphasise the persistence of these difficulties. Orton (1983) reports on the understanding of derivative held by one hundred and ten undergraduates in an introductory calculus course and states that these students showed some fundamental misconceptions such as confusion between: rate of a straight line versus rate of a curve; rate at a point versus rate over an interval; and the derivative at a point versus the point's y-coordinate. More recently, Rasmussen and King (2000) observe that the confusion between rate of a straight line versus rate of a curve, noted by Orton many years earlier, was still evident in the initial understandings of rate brought by students to their project. Similarly, Hassan, Mohamed, and Mitchelmore (2000) report that the students in their study had difficulty linking tangents to rate and derivatives. The twenty-seven tertiary Maldivian students in Hassan et al.'s study relied on formulae and rules. These students found it difficult to visualise the changing tangent as a point moves along a curve and were confused about the difference between average and instantaneous rate.

The demonstrated lack of conceptual understanding of introductory calculus limits its usefulness in related science applications. Lopez-Gay, Martinez-Torregrosa, Gras-Marti and Torregrosa (2002), in their study of 103 high school physics teachers and analysis of 38 Physics text books, stress the importance of students' understanding of differential calculus in understanding physics. They claim that physics students do not understand the use of calculus in simple real-world problems and have difficulty in applying it with autonomy. This suggests that the value of calculus to other fields of study is undermined by students' lack of conceptual understanding of it.

Despite the implementation of many innovations, such as the use of technology, designed to improve the outcomes of calculus courses, calculus students' lack of understanding of the fundamental ideas of change and rate persist (Hassan et al. 2000; Rasmussen & King, 2000; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Rowland & Jovanoski, 2004; Coe, 2007; Ubuz, 2007). Since these fundamental ideas of change and rate provide an important underpinning of derivative, researchers continue to persevere to find effective innovations. One such innovation is reported in this paper.

"[T]oday the concept of the derivative is usually presented first in calculus courses, with the notion of the integral coming later" (Boyer, 1970, p. 69). Currently, the usual sequence (Anton, Bivens & Davis, 2005 a recent tertiary introductory calculus text used in Australia) of introducing calculus involves limits, differentiation then integration where students are presented with a formal, abstract definition of limits and limit laws; a formal, abstract definition and rule ($dy/dx = \lim_{h \to 0} (f(x+h) - f(x))/h$) for differentiation; and integration viewed as anti-differentiation, with applications to area. In Anton et al. (2005), the first example of rate in their chapter on derivatives is the velocity of a moving body, such as a car or a ball. They emphasise this particular rate with a detailed discussion of displacement, velocity, average velocity, and instantaneous velocity. It is only after this detailed discussion that other examples of rate are mentioned, for example "the rate at which the length of a metal rod changes with temperature" (p. 153). This is followed by a definition for the slope of a linear function as "[a] 1-unit increase in x always produces an m-unit change in y" which is illustrated with the metal rod example. The unit rate is emphasised by the diagram seen in Figure 2. Variable rate is introduced through reference to the symbolic and graphic representations of the general function $y = f(x)$, again emphasising a unit rate approach (see Figure 2). The usual approach to the introduction of the concept of derivative assumes a sound understanding of rate and illustrates the derivative as the gradient of the tangent to the curve at a point, then moves quickly to emphasise symbolic manipulation.

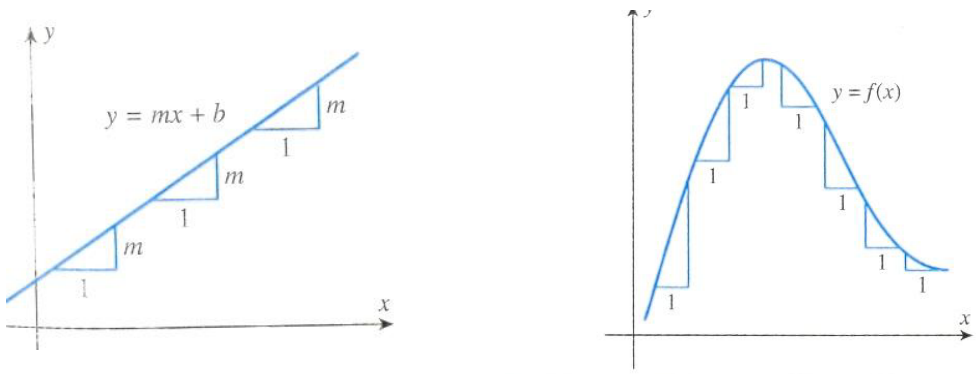


Figure 2. Rate diagrams from Anton et al., 2005, pp.153–154.

For example, in Garner et al. (2006), a mathematics text with an introduction to calculus used in some Victorian schools, the symbolic representations of a function are manipulated to establish a symbolic expression for instantaneous rate by taking the limit of the average rate. Some students become competent in this manipulation and can accurately produce the symbolic representation of the derivative (delos Santos & Thomas, 2005), but may not appreciate its meaning and connection to other mathematical concepts studied in earlier years.

Since rate is such a troublesome concept which affects the conceptual understanding of derivative, it is proposed that reversing the usual presentation of introductory calculus to begin with area and integration rather than rate and derivative, may improve students' conceptual understanding of this important area of mathematics. Doorman and van Maanen (2008) suggest that in "history we do not see the regular textbook approach from limits to differential quotient, from methods for differentiation to methods for integration, and finally the main theorem of calculus" (p. 10). This view is supported by Boyer (1970) who asserts "[t]hose textbooks that reverse the roles and place the integral before the derivative in a sense have history on their side, inasmuch as integration preceded differentiation by about two thousand years" (p. 69). This pilot study explores the effect of changing the usual sequence currently used to introduce calculus to begin with area and integration before rates and differentiation.

Method

Many mathematics students are introduced to calculus in secondary school; however, it was decided to trial the alternate sequence with tertiary students because of the high-stakes nature of the examinations of subjects undertaken in senior secondary school when calculus is usually introduced. It was anticipated that schools would be reluctant to participate unless they could be re-assured that their students would not be disadvantaged by undertaking the alternate sequence.

Two groups of first year tertiary students taking introductory calculus were selected to trial the effect of changing the traditional sequence described above. One group followed the traditional sequence and the other group followed a sequence beginning with integration. Both of these groups consisted of mathematics majors; human movement students intending to teach physical education and mathematics; and education students intending to teach mathematics. Whilst many students had been introduced to calculus at school, about one third of students had no previous experience of calculus.

Both groups were taught by the author—ensuring, as far as possible, a comparative delivery of the material. The learning experiences for both groups emphasised the support of a hand-held computer algebra system (CAS) and strong real-world connections. Delivery of content involved careful treatment of rate and extensive numerical integration stressing conceptual understanding as well as application of rules for differentiation and integration. At all stages of the delivery, strong, explicit connection to students' prior knowledge was attempted (Hiebert & Carpenter, 1992).

The concept of rate was explored in numeric, graphic and symbolic representations with instances of both constant and variable rate. The notion of average rate which built on students' understanding of linear functions, and hence constant rate, was employed to demonstrate a way of quantifying variable rate. The notion of instantaneous rate was developed by considering the average rates resulting from smaller and smaller intervals of the independent variable, that is, an informal treatment of the limit of average rate. The term instantaneous rate was eventually re-named derivative and symbolic manipulations undertaken to find the derivative from first principles using $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ noting that the fraction part is really just average rate. In this way, the concept of derivative was explicitly connected to the concept of rate. A consideration of limits was also employed in the introduction to numerical integration.

The concept of area was explored by considering a circle divided up into sectors (see Figure 1a) and finding the sum of the areas of the triangles bounded by two radii and a chord (the area shaded in Figure 1a). The number of sectors was increased and the table seen in Figure 1b completed, leading to a discussion of the relationship of this limiting process to the formula for the area of a circle. This exercise provided the link to students' prior knowledge for numerical integration. A feature of the introduction to the Fundamental Theorem of Calculus (FTC) was the explicit connection between numerical integration and differentiation from first principles (see Figure 1c).

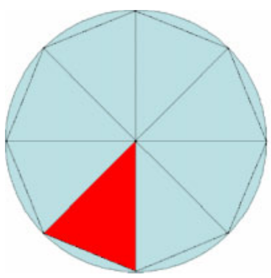
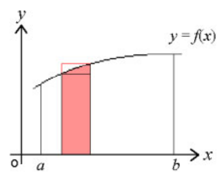


Figure 1a. Introduction to numerical integration.

No. of Triangles	Angle at centre	Total area
8	π/4	$2.82843r^2$
16	π/8	$3.06147r^2$
32	π/16	$3.12145r^2$
64		
128		

Figure 1b. Demonstration of limiting process leading to formula.



$$h \times f(x) \le A(x+h) - A(x) \le h \times f(x+h)$$

$$\text{As } h\to 0, \quad f(x) \le \lim_{h\to 0}\frac{A(x+h)-A(x)}{h} \le f(x)$$

$$\Rightarrow \quad f(x) = \lim_{h\to 0}\frac{A(x+h)-A(x)}{h}$$

Figure 1c. Development of FTC.

The results of the end of semester exam were compared using a two sample t-test. The exam included a mix of CAS-supported questions (see Figure 2, below) and questions which required conceptual understanding of the concepts of integration and differentiation (see Figure 3) and represented 50% of the assessment for the unit.

Your backyard pool is kidney shaped and its width can be modelled as a function of its length (x) using the rule $w(x) = -\frac{1}{324}x^4 + \frac{1}{9}x^3 - \frac{25}{18}x^2 + 7x, 0 < x < 18$ (a) Use a numerical method to the total area of the pool by finding the area under this graph, for example divide the interval up into four subintervals and add up the areas. (b) Check your approximation using a theoretical method.	Sketch one graph which satisfies all of the following conditions; $f(0) = 0$, $f'(-2) = f'(1) = f'(9) = 0$, $f''(x) > 0$ on $(-\infty, 0)$ and $(12,\infty)$, $f''(x) < 0$ on $(0, 6)$ and $(6, 12)$ $\lim_{x\to 6} f(x) = -\infty$ $\lim_{x\to\infty} f(x) = 0$
Figure 2. Example of CAS-supported question.	*Figure 3. Example of question requiring conceptual understanding of integration & differentiation.*

Results and discussion

Figure 4 shows the results of a two-sample t-test undertaken in Minitab on the examination results of the two groups. The p-value of 0.809 indicates that there is no significant difference between the examination results of the two groups at the 95% confidence level. This suggests that changing the sequence of delivery of introductory calculus had no effect on the overall performance of the students.

Descriptive Statistics: group 1 - trad, group 2 - alt

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Variable Mean StDev Minimum Q1 Median Q3 Maximum
group 1 - trad 64.19 22.84 14.29 50.89 68.45 79.76 98.81
group 2 - alt 62.76 19.13 20.67 46.33 60.00 79.67 96.00
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Two-Sample T-Test and CI: group 1 - trad, group 2 - alt

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Two-sample T for group 1 - trad vs group 2 - al
Difference = mu (group 1 - trad) - mu (group 2 - alt)
Estimate for difference: 1.43
T-Test of difference = 0 (vs not =): T-Value = 0.24 P-Value = 0.809 DF = 44
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Figure 4. Minitab printout of 2 sample t-test on examination results.

It was thought that greater insight into the results might be gained by separating the results for differentiation questions from integration questions. Figure 5 shows the results of a two-sample t-test undertaken in Minitab considering questions where an understanding of differentiation was required. The p-value of 0.857, seen in Figure 5, indicates that there is no significant difference between the examination results on the questions relating to differentiation of the two groups.

Descriptive Statistics: group 1 - trad-diff, group 2 - alt-diff

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Variable Mean StDev Minimum Q1 Median Q3 Maximum
group 1 - trad-diff 65.07 22.44 16.67 50.42 65.83 84.58 98.33
group 2 -alt - diff 63.98 20.64 22.22 47.78 65.56 81.11 94.44
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Two-Sample T-Test and CI: group 1 - trad-diff, group 2 - alt-diff

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Two-sample T for group 1 - trad-diff vs group 2 -alt - diff
Difference = mu (group 1 - trad-diff) - mu (group 2 -alt - diff)
Estimate for difference: 1.08
T-Test of difference = 0 (vs not =): T-Value = 0.18 P-Value = 0.857 DF = 47
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Figure 5. Minitab printout of 2 sample t-test on differentiation questions.

Figure 6 shows the results of a two-sample t-test undertaken in Minitab on the examination results of the two groups but only considering questions where an understanding of integration was required. The p-value of 0.064, seen in Figure 6, indicates that there is no significant difference between the examination results on the questions relating to integration of the two groups.

Descriptive Statistics: group 1 - trad-int, group 2 - alt-int

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Variable Mean StDev Minimum Q1 Median Q3 Maximum
group 1 - trad-int 61.98 28.40 0.00 41.67 66.67 83.33 100.00
group 2 -alt - int 47.84 25.15 0.00 28.13 50.00 68.75 93.75
```

Two-Sample T-Test and CI: group 1 - trad-int, group 2 - alt-int

```
Two-sample T for group 1 - trad-int vs group 2 -alt - int
Difference = mu (group 1 - trad-int) - mu (group 2 -alt - int)
Estimate for difference: 14.13
T-Test of difference = 0 (vs not =): T-Value = 1.90 P-Value = 0.064 DF = 46
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Figure 6. Minitab printout of 2 sample t-test on integration questions.

Conclusion

These results indicate that changing the sequence of delivery of introductory calculus was not detrimental to the development of conceptual understanding of introductory calculus for this group of tertiary students. However, many of the students in both groups had previously been introduced to calculus at school, so it is unclear how much this school-based introduction influenced their level of conceptual understanding. It may be that the alternate delivery trialled here with these tertiary students has a different effect when utilised at the school level. Perhaps this sequence of delivery may be more effective when all students are first introduced to calculus. This pilot study was undertaken to gauge whether the alternate sequence seriously disadvantaged students. This was an important consideration as calculus is included in the subjects taken by secondary school students in their high stakes examinations for university entrance. It was necessary to establish this before attempting to trial the alternate sequence at the secondary school level. Further research will be necessary to explore the efficacy of the alternate sequence for introductory calculus at the school level.

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