TEACHING LINEAR ALGEBRA: ONE LECTURER'S ENGAGEMENT WITH STUDENTS

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Linear algebra is a difficult introduction to advanced mathematical thinking for many students. In this paper we consider the teaching approach of an experienced lecturer as he attempts to engage his students with the key ideas embedded in a second course in linear algebra. We describe his approach in lectures and tutorials using visualisation and an emphasis on language to encourage conceptual thinking. We use Tall's framework of three worlds of mathematical thinking to reflect on the value of these activities. An analysis of students' attitudes to the course and their assessment results help to answer questions about the value of such an approach, suggesting ways forward in teaching linear algebra.

Introduction

Research examining the teaching of mathematics at university is a growing but relatively new field and, compared with school-based research, outputs are still relatively modest (Selden & Selden, 2001). Further, the research that has been conducted has rarely examined the daily teaching practice of mathematicians (Speer, Smith, & Horvath, 2010). Three possible reasons for this research lack are enunciated by Speer et al. (*ibid*) as: lecturing is a teaching practice rather than a common instructional activity in which teaching takes place; the professional culture of mathematicians tends to obscure differences in teaching; and strong content knowledge, well-structured for students is considered sufficient for good teaching. Among studies that have been conducted, Rowland (2009) documents the way a university teacher's beliefs about mathematics led her to implement changes to her style of teaching, avoiding a paradigm of exposition and note-taking. Instead she introduced an interactive environment in which class session exercises, testing of conjectures and sense-making were commonplace. Establishing such a community of inquiry in any classroom requires all involved to believe that all participants are learners (Jaworski, 2003). Another study, focussing on university linear algebra teaching (Jaworski, Treffert-Thomas & Bartsch, 2009), examined teaching from a community of practice perspective. This research highlighted that how to deal with the common difficulty of a didactic tension between an abstract/conceptual approach and one that emphasises computational facility is not well understood at university. We agree with the authors that: "Awareness of didactical challenge and a didactic tension can illuminate practice more broadly." (*ibid*, p. 256) and that doing so through a community of inquiry is likely to be a productive way forward. The research described in this paper investigated, through a community of inquiry, how linear algebra may be taught to promote both procedural and conceptual understanding and thinking. Linear algebra demands a more formal approach than calculus, making it difficult for undergraduates to understand the subject (Dorier & Sierpinska, 2001) and research suggests that many students have a minimal understanding of concepts, manipulating matrices instead to pass examinations.

Theoretical framework

Tall's (2004, 2008, 2010) developing theory of three worlds of mathematical thinking seems highly relevant for analysing linear algebra students' thinking processes. It introduces a framework for development of mathematical thinking based on three mental worlds of mathematics: *conceptual embodiment*; *operational symbolism*; and *axiomatic formalism* (Tall, 2010). The embodied world is enactive and visual. It contains embodied objects; it is where we think about the physical world, using " ... not only our mental perceptions of real-world objects, but also our internal conceptions that involve visuo-spatial imagery" (Tall, 2004, p. 30). The symbolic world is the world of *procepts*, where actions, processes and their corresponding objects are realized and symbolized, and the formal world comprises defined objects (Tall, Thomas, Davis, Gray & Simpson, 2000), presented in terms of their properties, with new properties deduced from objects by formal proof. All three worlds are available to, and used by, individuals as they engage with mathematical thinking. In particular, the three worlds of mathematical thinking combine so that "three interrelated sequences of development blend together to build a full range of thinking" (Tall, 2008, p. 3). A pedagogical implication is that the framework is not proscriptive, and "(a)lthough embodiment starts earlier than operational symbolism, and formalism occurs much later still, when all three possibilities are available at university level, the framework says nothing about the sequence in which teaching should occur" (Tall, 2010, p. 22). For example, Tall claims that many students learning mathematical analysis are happy to think and operate entirely in the formal world, whereas others prefer a more natural approach and think in terms of thought experiments and concept imagery. Thus no single approach is privileged over another, instead decisions should be based on the objective of each course "and not to inflict formal subtleties on students who are better served by a meaningful blend of embodiment and symbolism" (Tall, 2010, p. 25).

Method

The research reported here employed a mixed-methods approach, partly an action research project in which the first-named author (referred to as 'the lecturer' or 'John' in what follows) worked with the other authors as he tried to determine the effectiveness of certain aspects of his teaching of introductory linear algebra, forming a community of inquiry to discuss the teaching openly, and partly a case study of the students. The project involves cycles of planning the relevant teaching episodes, implementing them, and then reflecting on and evaluating the results. Data was collected in 2010 from a second year linear algebra course at the University of Canterbury taught by the lecturer. About 170 students took the course, almost all of them majoring in science or engineering. The lecturer was interviewed (in a discussion mode) twice by the other two researchers, in connection with each of the stages of the project. Thus some of the discussion focussed on his overall goals for the course, and in particular to determine how these goals related to Tall's framework of three worlds (embodied, symbolic and formal) of mathematical thinking, or to the relationship between language and understanding. In addition, other questions dealt with the day-to-day implementation of these goals during lectures and tutorials (the lecturer kept a diary of what happened after each class). Finally, some questions dealt with how the course was measuring up against the intended goals. There were also regular Skype discussions after the lectures had finished as part of the community's discussion of these issues. The interviews and the Skype discussions were audio-recorded and later transcribed for analysis. Data for the case study providing a student perspective comes from several sources. A good number of students (48 out of 170) allowed us to examine their responses to test questions, some of which had been designed to elicit information about their acquisition of the language of linear algebra, or about their relationship to the kinds of thinking described by Tall's framework. About 100 students filled in a survey about the lecturer's teaching and three-quarters of these gave responses to some open-ended questions. Finally, a small number (nine) of students volunteered for individual semi-structured interviews two weeks after the completion of the course. They were asked about definitions (Can you give me the definition for any of these terms in the first question? Were you confident with the definitions during the course?), geometry (Which of these terms in part A can you describe geometrically? Would geometry help you to understand it better?) and general questions (How did you find linear algebra in general? How did you learn the concepts?). Some of the points that came up in discussion with the lecturer were: I wondered if you'd like to tell us how you see the role of the tutorial; I think you like to get the definition motivated by what you're doing in solving equations. Can you tell us how that works?; What's your view of the use of technology in general in this course; How confident were the students in speaking the linear algebra language?; What was your thinking behind setting the exam questions?

Results

In this paper we principally describe the outcomes in terms of the students, and their reactions to the style of the course and evidence of their consequent learning of linear algebra in relation to the lecturer's expectations. The course was founded on the value of language, visualisation, technology (Matlab) and writing and problem solving in tutorials to give students the tools to think about mathematics for themselves. This was all part of what John called trying to put across the "big picture". Both the lectures and the tutorials had to fit in to this overarching aim, and an example of a section of a tutorial, to show the general philosophy behind them, is given in Figure 1.

The tutorial exercises look at span and linear independence for typical vectors in 2-space and 3-space, and also look at the geometric meaning of span and linear independence.

1. (a) i. Let $\mathbf{u}_1$, $\mathbf{u}_2$ be two vectors in 2-space. Does $\mathbf{u}_2$ usually belong to the span of $\mathbf{u}_1$? *Hint:* Use Matlab's rand command to construct random pairs of vectors. Use rref if you need to solve any systems of linear equations. ii. Does $\mathbf{u}_2$ always belong to the span of $\mathbf{u}_1$? Give an example of each possibility. iii. Interpret your results geometrically ... [Formatting changed]

To be handed in: Write a short report (at most one side of A4 paper) describing your results. Your report should consist entirely of English sentences, with no symbols or equations.

Figure 1. An example of a section from one of the tutorials.

The overall aims of the tutorials were expressed to the students by the lecturer as: Learn the technical terms used in linear algebra; Get a feel for what usually happens in linear algebra, but be aware of exceptions; and Be able to describe what happens in linear algebra using ordinary English. As John said in his first interview:

> The main reason I'm in the job is I like helping people. I'm curious about how people think, why they do the things that they're doing. I'd like to show them other ways of thinking, but I'd really prefer that they went out into the world thinking for themselves, and if I could give them some tools that will do that, that will be really nice.

The student interviews were revealing about the lecturer and student perspectives on each of the areas of language use in tutorials, definitions, and visualisation, and these are considered below.

An emphasis on language

For this year's version of the course, John decided to put greater emphasis on gaining the "big picture" through getting the students to use and understand the language of linear algebra. Some of his motivation for this surfaced while he was trying to use a theoretical framework to analyse an incident during lectures where he had been telling the class about how the tutorials were going to work "[My] actual goal [here] is getting students to use written language to describe mathematical ideas, events, etc. because I think this will help them to learn or understand the new ideas." In order to promote this aspect of writing about mathematical ideas, John put some thought into what the tutorials for the course should be like. What resulted was that one aspect of the tutorials required students to write about their ideas. For example, in Figure 1 we see the direction in the tutorial to 'Write a short report'. John's reason for this was that "they're not used to being asked questions like ... 'write a paragraph of 75 words about such and such' ... a really common response to that was to just write down all the relevant definitions in sequence, and not make any reference to what they were actually asked for." Clearly focussing on ideas and language, devoting time to experiments and reports, comes at a cost. John expressed how "I've sacrificed tutorial time that would normally be spent doing hand calculations ... I've told the students, 'Well, actually you can do that in your own time. There's a consultancy session where you can go and get help if you're stuck. But I want to use the tutorial to do something extra." When he reflected on the value of the tutorials John observed that the students found it hard to express themselves mathematically in written language. However, in spite of their struggles at times, they were attending the tutorials in greater numbers than previous years and were more active participants.

> They're certainly behaving very differently from last year's class. For a start, even though there's no compulsion on them showing up to tutorials, I've got I'd say two thirds of the class actually showing up to the tutorials, whereas last year we were lucky if we got a third of them coming along ... the talking in the tutorial is definitely different as well.

Overall the students who were interviewed were often positive about the tutorials.

> S2 I guess formal reports is pretty good ... talking about it with someone else is actually really helpful, so it depends, I would probably keep that ... we had to write reports on certain questions ... and I think that was a really good way of learning the definitions and applying them.

S4 The tutorial system I thought was exceptional. Because ... we had tutorials once every two weeks and even though ... you really had to think about them. So I kind of developed the ideas quite a lot in my head.

S5 The tutorials were quite helpful because you go through and it says what did you learn, and you learn something by doing it.

Formal world thinking often begins with definitions of objects (Vinner, 1991) and the students were given a list of definitions from the start of the course; they were not expected to learn them but to talk and write about them. In fact they could take an A4 sheet into the examination with data, including definitions, written on it. The students' responses to this approach showed that they understood the importance of the language and the need to be able to talk about the ideas.

S6 Yeah, to the subject, they [definitions] are quite important, cause much of that area, linear algebra can't be described without actually understanding and knowing those terms.

S8 Yep, yep, it was made pretty clear to us that these were terms that we were going to need to know, in and out, and we were going to be able to have to use them in conversation ... So it was yeah made pretty clear that they were going to be very, very important ... They [the definitions] were definitely taught.

Visualisation

One of the cornerstones of John's approach to teaching was the value of visual imagery, in terms of encouraging mental imagery through the use of both physical objects and pictures. This is related to the embodied world of Tall's framework, which involves iconic and enactive actions. In the lectures John employed a combination of embodied, iconic and enactive, physical ideas with props, as well as pictures, to get across the ideas. He also values being able to make links between the representations. In his reflections on the lectures he indicated the value of pictures to him personally "I think I have always liked a good picture, although I don't remember any pictures being used when I learned linear algebra as a student—it was just lots of calculations, usually row operations." However, he is conscious of the need " ... to strike a balance between what my colleagues want the students to know in later courses (usually technical stuff like 'how to do this type of calculation') and giving them ... pictures or 'what this all really means', or ... communication skills." Some of the physical, enactive demonstrations he used, and the fact that a picture was also drawn, were described in his lecture reflections:

> I assembled a solid picture of our problem with the rectangular piece of board as the subspace U, my red OHP pen standing on end to represent the given point v (at the top of the pen) so that the projection p that we seek is at the base of the pen. A picture version of the situation was drawn too.
>
> So I waved my board and a pointer, and then drew a picture, illustrating that if our plane W went through the origin, then the plane and (a suitably positioned) normal line U were both subspaces of 3-space, and that vectors chosen, one from each subspace, were always perpendicular.

However, the pictures were sometimes used to show mathematical relationships, as seen in this example, which refers to the picture in Figure 2:

> I decided to remind them of our earlier picture of the action of a 2x2 matrix A [see Figure 2]. We see now that what was called the 'range of the transformation given by A' is actually col(A) and what was called the 'solution to Ax=0' is actually null(A). The other

line in that diagram is not a subspace as it does not go through the origin (or zero vector). But our third subspace, row(A) can be pictured in this diagram too ... vectors in row(A) are all perpendicular to vectors in null(A), so we can add a line representing row(A) to the domain part of the diagram, perpendicular to the line representing null(A).

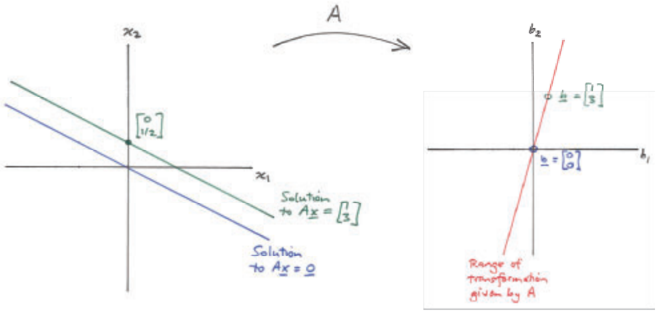

Figure 2. A picture of the effects of a linear transformation.

The interviews with the students showed that they valued the imagery, both enactive and pictorial, that John had incorporated into his explanations.

S1 We did lots and lots of drawings about taking the vector away and when he was describing linearly independent he did some quite good visualisations as well ... He actually got like two sticks or whatever and dropped them and said they're not parallel these are linearly independent.

S4 Yeah John did a lot, yeah he had a lot of using rulers and pencils. I really enjoyed it. I thought it was a fantastically taught class ... he just had a very visual emphasis in the class and really helpful.

S9 Yeah definitely, it definitely helped seeing the pictures. If I'd just read that, I wouldn't have got any kind of a grasp ... I might have been able to do some of it [without the pictures], but I definitely wouldn't have been able to do it as well.

In the interviews students talked about some of the geometric images they used to understand constructs: "Yeah linear combination ... I visualise that parallelogram when you add the vectors." (S3); "The span of two, one of the independent vectors no matter what space it's going to be a plane so if that's going to be a plane ... using a multiple of each of the spanning vectors you can get to any point in that sub space." (S4); and "So linear combination of a couple of vectors is going to span out the plane unless they're along the same line in which case they're dependent so just spans out that line." (S7).

The outcomes

Of course innovation in one's teaching does not necessarily imply that it has value for student understanding. Hence we have to ask the question, did it help understanding? John's answer to this includes the statement:

> You've got to remember, there were almost 170 students in the class ... I did feel that they had a better grasp of things than they've had in previous years ... There were some really pleasing ones, and on the exam I had that question where, I gave them three vectors and they had to talk about them, basically write a little story about them, using all the words that we learnt, and there were some really nice answers to that.

Certainly the student evaluation of the course supported the view that the students liked the approach taken. The scores provided by the 101 respondents were: Q1 The classes were well organised 4.7(/5); Q2 The lecturer was able to communicate ideas and information clearly 4.5; Q3 The lecturer stimulated my interest in the subject 4.1; Q4 The lecturer's attitude towards students was good, 4.7; and Q5 Overall, the lecturer is an effective teacher, 4.6. In addition, some of the comments made in the open question on the evaluation were often extremely positive. Examples include: "Best lecturer I ever had in Engineering" (S1); "One of the best lecturers in [name] university. Asset!!" (S8); " ... this course is probably one of the best taught classes I've had" (S20); "Excellent lecturer—Interesting presentations ... Examples and physical representations useful" (S37); and "I think he has perfected the art of teaching. His teaching style matches my learning style. Keep up what he is doing and students will do well." (S44).

Was this positive view of the course borne out by the assessment results? In the mid-semester test, in addition to standard skills, a conceptual question was included to examine the students' ability to relate mathematical thinking across the embodied, symbolic and formal worlds. Question 3(a) had two parts as seen in Figure 3. The mean score on this question was 6.55 out of 12, and the students did significantly worse on this question than on question 1, which comprised standard procedures (mean_{Q1}=60.4%, mean_{Q3}=54.6%, t=2.98, p<0.005). However, given the testing nature of some of question 3 this is a reasonable result. In part (a)(i) of the question they were required to interpret the symbolic-algebra equation **u** = 2**v** + 3**w** in an embodied-process manner by drawing a diagram. 67% correctly drew either a parallelogram or a triangle to represent the vectors and a further 27% were partly correct.

(a) Suppose that **u**, **v**, and **w** are nonzero vectors in $\mathbb{R}^3$ such that $\mathbf{u} = 2\mathbf{v} + 3\mathbf{w}$. i. Draw a diagram to illustrate the relationship between **u**, **v**, and **w.** ii. Use the appropriate technical terms from linear algebra to describe the relationship between **u**, **v**, and **w.**

(c) Suppose that **u**, **v** are linearly independent vectors in $\mathbb{R}^3$. i. Give a geometric description of the span of **u** and **v.** ii. Which of the following sets of vectors could be a basis for $\mathbb{R}^3$? (α) **u**, **v**, **u** + 2**v**. (β) **u**, **v**, **u**□**v**. (γ) **u**, **v**, **u** + 2**v**, **u**□**v**. [Formatting changed]

Figure 3. Two parts of the test question 3.

Part (a)(ii) then asked them to use technical terms to describe the relationship between **u**, **v** and **w**. Of the 35 students who got full marks on this part, 5 students mentioned only one concept, namely linear combination, 26 students mentioned 2 concepts (and 18 of these spoke of linear combination and span), and 4 students mentioned 3 concepts (linear combination, span, and linear dependence). Typical examples of comments from students in these three groups were: "**u** is a linear combination of **v** and **w**"; **u** belongs to the span of **v** and **w**"; and "**u**, **v** and **w** are linearly dependent". Part c(i) examined whether students could relate the definition of span of two linearly independent vectors to an embodied process. Tewnty-seven were able to say that the span was a plane in $\mathbb{R}^3$, but only 10 gained full marks by going on to say that both **u** and **v** would lie in the plane. Only seven of the students drew a picture for this part. For c(ii) the students needed to understand the definition of basis and then be able to test whether the sets of vectors satisfied the conditions that the set must a) be a minimum spanning (or generating) set and b) comprise linearly independent vectors, testing the relationship between the formal world definition of basis and symbolic-algebra object thinking. Only one drew a picture, showing that embodied thinking was not to the forefront on this question, and two used matrices to assist them, showing an absence of symbolic-matrix thinking. Some excellent thinking and reasoning targeted the key properties (see Figure 4).

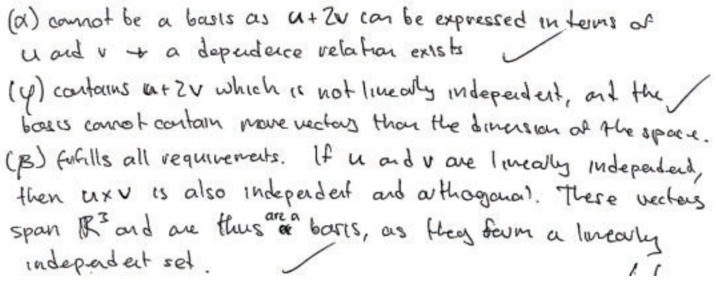
(α) cannot be a basis as u+2v can be expressed in terms of u and v ∴ a dependence relation exists
(γ) contains u+2v which is not linearly independent, and the basis cannot contain more vectors than the dimension of the space.
(β) fulfills all requirements. If u and v are linearly independent, then u×v is also independent and orthogonal. These vectors span $\mathbb{R}^3$ and are thus ~~a~~ are a basis, as they form a linearly independent set.

Figure 4. An example of clear application of the properties of basis.

It was also interesting that many used informal ways of thinking about the required properties. Examples of this included the vectors in (γ) being rejected because the set is ‘not efficient’ or there is a ‘redundant’ vector. In this latter case they were echoing John’s language, since in his lecture commentary he used it repeatedly, and wrote down the working definition of basis he gave them as: “A basis for a subspace is a set of vectors which span the subspace and in which there are no redundant vectors”, avoiding use of the term linear independence.

The final examination was a traditional one covering the whole course, with the required test of skills, such as using the Gram-Schmidt process to find an orthogonal basis for the column space of a given 4x3 matrix (Q2(b)). However, it also included questions such as “3(b) Give a geometric description of the following situation: **a, b, c** are linearly independent vectors in $\mathbb{R}^3$.” and 3(c): “Consider the vectors **u** = (1, 0, 0), **v** = (0, 2, 0), **w** = (3, 4, 0) [given as columns]. Write a short paragraph about **u**, **v** and **w**. Your paragraph should be at most 75 words long, but should include as many as possible of the following technical terms from Linear Algebra: *basis, dimension, dependence relation, linear combination, linearly dependent, linearly independent, span, subspace*.” Once again students found such questions harder than the more algorithmic questions. The average mark for Q2(b) was 2.6/3 whereas the averages for Q3(b), (c) were 0.8/2 and 3.8/6. However, a significant number of students gave fully correct answers (e.g., 22 out of 162 students got 6/6 for Q3(c)). The distribution of the final examination marks showed a mean mark of 32.7 out of 50, with 12.3% above 40 and a pass rate of 89.5%. These results compare favourably with previous years’, so the students were not disadvantaged in traditional understanding by the course presentation.

Conclusion

In this study we have looked at student reactions to a particular style of delivery for a second year course in linear algebra, and at the effect this style may have had on the students’ learning. Features of the delivery were the emphasis on language, visualisation and experimentation using technology. Experimentation was structured into fortnightly tutorial sessions, visualisation was encouraged through use of models and pictures in lectures, and language was emphasised in report writing. Students were generally positive about all these features of the course. Talking to fellow students during the experiments, knowing the correct technical language and actually using it in written reports, having to think about the material—all these things were reported as having helped them to learn. This reaction may not seem very surprising, but it is perhaps little unusual coming from a course in linear algebra. Students were also positive about the use of visualisation. For some, memories of boards and marker pens being manipulated conjured up the process of finding projections, while for others, sticks being dropped on the floor remained vivid reminders of linear independence. The assessment results show that many students found these ideas quite challenging. Most students performed better on routine algorithmic tasks in the test and exam, than they did on tasks exploring links between the geometric, symbolic and formal views of linear algebra. This is hardly surprising, of course, as algorithms can be applied without understanding, whereas the other tasks require making links between different representations of the concepts. A pleasing feature of the study was the number of students who succeeded in writing coherent prose that linked the various concepts of linear algebra, to each other in the formal world, and to concrete visualisations in the embodied world. On reflection the lecturer feels quite pleased with the result of his pedagogical experiment, and the supportive community of inquiry worked well too. A good number of students in a second year class learned how to express themselves in the language of linear algebra without loss of skills. This isn't to suggest that we have solved the problem of how to teach linear algebra, since some concepts, such as linear independence and basis, seem harder to learn than others. There lies the continuing challenge.

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