MODELS OF MODELLING: IS THERE A FIRST AMONG EQUALS?

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Given the variety existing within mathematical modelling enterprises, it is not surprising that different perspectives have found their way into educational practice and research. A variety of genres, (or variations within genres), has emerged within education communities, who use the term 'mathematical modelling' with different emphases, and in some cases with different meanings. This presentation will review the origins and purposes of several articulations of mathematical modelling. Tensions will be identified, and some inconsistencies and misplaced inferences illustrated. Different approaches will be linked to underlying purposes that are not always made explicit, and some specific issues will be highlighted.

Introduction

Mathematical Modelling, its practice, research, and curricular implications continue to engage members of the mathematical and mathematics education communities. In Australasia recent foci are found in the *MERGA Review of Research (2004—2007),* in the recent special issue of the *Mathematics Education Research Journal* (Stillman, Brown, & Galbraith, 2010), and through the ongoing published work of individuals.

Practitioners and researchers inhabit different sections of the respective communities, as well as the interface between the two. Hence it is not unexpected that different perspectives share similar terminology when talking and writing within the field—resulting in a variety of genres, and variations within genres among those who use the term 'mathematical modelling'. Confusion is generated when individuals lay a particular meaning over writings and other scholarly products that have been constructed within a different genre, while more fundamentally, value judgments concerning the purposes and features of application and modelling initiatives stand to be distorted by generalisations made on the basis of limited experience or understanding, or indeed selective referencing.

Structure and purpose

This paper first reviews the characteristics of several articulations of mathematical modelling and applications as found within the mathematics education community. Its lens focuses on mathematical modelling as it interacts with curricular purposes within mathematics education, rather than analysing particular variations viewed from within the modelling field—as in Kaiser & Sriraman (2006). Some criticisms of mathematical modelling will be illustrated, tensions identified, and inconsistencies and misplaced inferences illustrated. Different approaches will be linked to underlying purposes that are not always made explicit, and some specific issues highlighted. Finally reference is made to a stated aim of the proposed Australian Curriculum—Mathematics (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2010): that "mathematics aims to ensure that students are confident, creative users and communicators of mathematics, able to investigate, represent and interpret situations in their personal and work lives and as active citizens".

It will be argued that at most two contemporary approaches that use the term 'mathematical modelling' can hope to contribute decisively to such an aim.

Models of modelling

For present purposes the focus is on terms and meanings associated with 'mathematical modelling' that are recognised within the international community of practitioners and researchers in the field (Blum et al., 2007). Six approaches to the use of mathematics with connections to the real world are considered below.

1. Using real problem situations as a preliminary basis for abstraction

Two studies that used practical contexts to motivate and develop the linear relationship (y = mx + c) at respectively years 9 and 8 levels, are reviewed in (Bardini & Stacey, 2006; Bardini, Pierce, & Stacey, 2004). Symbolic, numerical, and graphical representations of the relationship were introduced by considering the cost of hiring trades people, where the given conditions included a flat 'call charge', together with labour charges on a per-hour basis. With the year 8 above average ability students, graphical calculators were introduced to facilitate the learning. Axial intercepts, slopes, points of intersection, and intervals required interpretation in context, across a variety of problem settings. The students learned to write algebraic rules in conventional formats, were comfortable selecting symbols that made sense in terms of the problem settings, and showed understanding of the function property of expressing one variable quantity in terms of another. Problematic was the time factor—five weeks seems a very heavy investment for the achieved outcomes. Since the approach had to cater for a pre and post testing format, perceived clashes between research requirements, and authenticity of problem solving were resolved at the expense of the latter. For example students made decisions about contextualised problems on their own, where in reality a decision about which plumber to hire would usually be a collaborative (e.g. family) decision reached after some discussion of competing quotes. This is all about the team nature of aspects of a modelling process, whose goal is to obtain and justify the solution to a problem. Some useful outcomes were achieved in both studies—the time commitment was problematical, and expedient rather than authentic modelling practices were imposed at times.

2. Emergent modelling

Emergent modelling (Gravemeijer, 2007; Doorman & Gravemeijer, 2009) is an instructional design heuristic, developed as a component of a domain-specific instruction theory generated within the Reality in Mathematics Education framework in the Netherlands. ‘Emergent’ refers both to the nature of the process by which models emerge from students’ experience, and to the process by which these models support the emergence of formal mathematical ways of knowing - that are no longer dependent on the support of the original models. That is, there is emphasis on a search for models that can be developed into entities of their own, and subsequently into models for mathematical reasoning. Gravemeijer (2007) summarises the process as one of “abstraction-as-construction” in which mathematical knowledge is grounded in earlier experiences that are meaningful and applicable. In that they are familiarised with a mathematical take on everyday life situations in the process, students are incidentally prepared for more serious application and modelling adventures in the future - indeed Gravemeijer has referred to emergent modelling as a precursor to mathematical modelling. Emergent modelling can also be viewed as a more organised and theorised approach than that described in the previous section, which typified approaches aimed at using contextualised mathematics to motivate and attain proficiency with the form of a basic mathematical relationship.

3. Modelling as curve fitting

This approach has become increasingly significant with the availability of regression menus in software and graphical calculators. A model generated by this means can become a purely technical artefact whose parameters vary with the particular data set, and which can be generated in complete ignorance of the principles underlying the real situation—indeed undertaken without knowledge of where a table of data comes from. It raises a profound theoretical issue—the relative authority of data as such, versus the theoretical structure underpinning its generation In one example curves were fitted to population data by using successively the full suite of regression choices available on a graphical calculator—with no apparent realisation that data generated by births deaths and migration should have an underlying exponential pattern. Curve fitting remains an important activity within the modelling enterprise, but when used mindlessly it creates a dangerous aberration of the modelling concept. Riede (2003) demonstrates good modelling practice when relating weightlifting records to weights of athletes. An inverted parabola was postulated to model the data, on the grounds that weight lifted at first increases with body weight, but ultimately (beyond the super heavyweight class) begins to decrease as body weight impairs the ability to lift. The subsequent fit was excellent.

4. Word problems

Vershaffel (e.g. Greer & Vershaffel, 2007; Vershaffel &Van Dooren, 2010), has been writing and researching insightfully, for many years on the subject of student approaches to word problems. Studies in a variety of countries have consistently demonstrated the propensity of students to ignore contextual factors, and apply (often incorrect) actions based on perceptions of what school mathematics is about—such as being divorced from reality. His work with colleagues has included a focus on the suspension of sense making by students while working on word problems, so that aberrations are produced that the same students would never contemplate in their real lives outside the classroom. Various intervention studies to identify problems and stimulate improvement have been designed and implemented, with varied outcomes (see Vershaffel & Van Dooren (2010) for a summary of some of these). Attention is drawn to the impact of classroom culture in seeking change, for not only are different types of problems needed, but improvement "would also imply a classroom culture radically different from that which typically exists in many mathematics classes." The difficulty of producing change may well be compounded by the types of intervention materials proposed—more realistic word problems in text books do not address the cultural issues that learning from text books in this area themselves reinforce. If the medium remains the same a different message is difficult to promote.

5. Modelling as a vehicle for teaching other mathematical material

When used as a *vehicle* (Julie, 2002), modelling contexts are chosen so that the mathematics of interest is embedded in the associated examples. The principal driving force in defining the boundary of the activity is curriculum content, and genuineness of applications is made subservient when necessary (not always), to achieve this perceived need. This particular emphasis is most clearly expressed by Zbiek & Conner (2006).

> … we recognise that extensive student engagement in classroom modeling activities is essential in mathematics instruction only if modeling provides our students with significant opportunities to develop deeper and stronger understanding of curricular mathematics. (pp. 89–90)

These authors describe the implementation of a problem involving the siting of a hospital to service three large (actual) cities in north western USA, so the data were real. The subjects (student teachers) were left to address the problem in their own way, and later interviewed to probe their use of assumptions, strategies, parameters, interpretations, and justifications. All these aspects are involved in real world modelling, except here it was the evocation of these separate mathematical and problem solving entities as such that was of central interest. In fact the authors' did use a modelling process that was included in the paper but not shared with the students.

English (2010) also working within the *vehicle* mode, used environmental material as a stimulus for engendering data based modelling involving classification and display of attributes, with first grade children. We are reminded that, as has been pointed out many times, the capacity to learn from modelling examples is not a function of age or the amount of mathematics that is known—although the types of modelling activities that are suitable clearly are impacted by experience and knowledge.

While the approaches described within 1 and 2 (above) also use real contexts, this genre is much more thorough in using a modelling process to generate information of value, even if (as with Zbiek & Conner, 2006) the students are not made aware of this. Lesh and associates (e.g. Lesh & Doerr, 2003), use carefully constructed Model Eliciting Activities (MEAs) to elicit mathematical concepts for consolidation and enhancement.

6. Modelling as real world problem solving

This perspective differs in some important respects from those discussed so far, firstly because its origins lie substantially with those who have used mathematics to model problems in professional fields outside education, and in their personal lives. Some, such as Pollak (telecommunications), Burkhardt (physics), and other early ICTMA progenitors, have taken their experience and insights specifically across into modelling initiatives in education, while others (e.g., Pedley, 2005) without specific intention to do so, continue to provide external reference criteria for those working within the educational field. An essential goal is for students to develop and apply modelling skills to obtain mathematically productive outcomes for problems in their world with genuine real-world connections. Modelling in this vein has two concurrent purposes—to solve a particular problem at hand, but over time to develop modelling skills, that empower individuals to apply to problems in their world. That is, to become productive users of their curricular mathematical knowledge. Characteristic of this approach is a cyclical modelling process—containing elements such as the following drawn from the Presidential address to the *Institute of Mathematics and its Applications,* (Pedley, 2005). Note that Pedley was not addressing an audience of educators.

Understand the real problem situation.
↓
Frame an appropriate mathematical question
↓
Formulate a model, using simplifying assumptions etc
↓
Analyse the model
↓
Compare mathematical outcomes with reality
↓
Modify and repeat until an adequate solution has been found.

The arrows on the right indicate that iterative back tracking may occur between any phases of the modelling cycle when a need is identified. This diagrammatic translation of Pedley's message is a compact version of the modelling chart that appears in various representations in many sources (e.g., Galbraith & Stillman, 2006). Such diagrams describe the modelling process, but also act as a scaffolding aid for individuals or groups as they develop modelling skills through successive applications. The labels do not represent vacuous generic properties, but attributes that find specific and different instantiations, depending on the context in which the modelling takes place. In this genre the solution to a problem must take seriously the context outside the mathematics classroom within which it is introduced, and its evaluation involves returning to that context. It cannot live entirely in a classroom.

It is argued that two substantive strands can be identified that contribute to this approach to modelling. One of these involves using MEAs (Lesh & Doerr, 2003) as 'modelling development tutors' when used as orchestrated activities within a carefully planned sequence. When used thus, the identification of relevant mathematics with which to model becomes a central feature, an aspect that is reduced when MEAs are used in close proximity to a cognate topic to elicit or consolidate particular concepts—where their purpose is more often in the vehicle mode. Note that this amounts to creating a strength out of the use of MEAs as 'stand alone activities', perceived as a weakness by Yoon et.al., 2010 when used in isolated and uncritical ways.

The other strand typifies the emphasis of the ICTMA group (e.g. Blum & Leiss, 2007), in which a modelling chart is used as a scaffolding aid with dual purpose. One purpose is to assure that real world problem solving in education contains the procedures, and checks and balances that professional mathematical modellers endorse and apply. The second purpose is to provide a means for individuals to build and test modelling expertise, including use of curricular mathematics, through engagement with a selection of appropriately chosen problems, including at times their own. The emphasis is on learning how to identify problems, and to formulate related mathematical questions that can be addressed with existing mathematical knowledge—developing this ability is one of the most significant challenges new modellers face.

Critique of modelling

This section selects and reflects on some criticisms made regarding mathematical modelling. Jablonka & Gellert (2007) argue that there is no straightforward way to move from a real problem context to a mathematical model, because it is virtually impossible to quantify non-mathematical characteristics, and relate them mathematically in *one* step. There is confusion here between a procedure (step) and a phase in the modelling process—the latter may contain several steps and will vary in complexity with the sophistication of the problem. They further argue that there can be no *validation* because a result is not put back into a 'real' *real situation*. We will return to this criticism in the concluding section.

Arleback (2009), when introducing modelling to upper secondary students, could find no evidence of the cyclic process widely described in modelling research: "the discrepancy with what actually happens is palpable". This was a strange observation, as shortly before he had identified sub-processes that characterised the students' work: reading; making a model (structuring and mathematising); calculating; validating; and writing. All these are essential elements of the modelling process, and validating cannot occur without reviewing a solution in terms of the original problem statement (reading). This alone completes a cycle, even without further cycles introduced through the checking and reviewing that inevitably takes a solver back through earlier phases in producing a defensible solution. This comment is enigmatic, as the paper in general is carefully constructed, and the description of the research is excellent.

Sfard (2008) claims that, the minute an 'out of school' problem is treated in school it is no longer an 'out of school problem', and hence the search for authentic real world problems is necessarily in vain. There are several examples in the literature where individual students have, on their own initiative, used mathematical modelling techniques learned in school, to address situations in their personal lives outside school—as authentic as one could wish. Again we will return to this point in the final section.

In a similar vein Barbosa (2006), argued that "since students and professional modellers share different conditions and interests, the practices conducted by them are different." While there are differences of course, what both groups need for success are modelling competencies that can be applied effectively and sensitively, including the ability to work productively both as individuals and as team members. The following questions are relevant for both groups. Is it important to be able to: Define a problem from a real-world setting? Formulate and defend an appropriate mathematical model to address it? Solve the mathematics involved in the model? Interpret the mathematical results in terms of their real world meanings and implications? Evaluate and report the outcomes of the model both for mathematical validity, and in terms of their relevance to the original question? Revisit and challenge material produced within any part of the modelling process in the interests of improved outcomes? Can any of these 'stages' be omitted from a seriously constructed modelling endeavour? Is the ordering of the stages arbitrary? If as we contend the answer to every question except the last two is "yes", and to those two is "no", we have a process that characterises essential modelling activity that is as relevant to school learners as it is to those doing modelling professionally or for personal reasons.

In a very recent paper Jablonka & Gellert (2011) begin with the assertion that "Modelling approaches are propagated to enhance the quality of the outcomes of mathematics education by providing students with generic competencies and thereby creating a flexible work force". This is a sweeping and mistaken generalisation, as motivations are various, and include centrally that of student empowerment, as in: "… for students to spend years learning mathematics without any sense of how to apply it in the world around them, is inappropriate" (Stillman, Brown, & Galbraith, 2010). The paper is a mixture of observations, assertions, and arguments. It raises some important issues concerning equity, but a drawback is the dependency on selections chosen seemingly to support the ideology of the critique, rather than a representative spectrum from the field. For example, the authors allege that modelling conceptions do not see associated competencies as 'culture bound and value driven." Yet an introduction in Blum et al., (2007) points out that "the best route for a new freeway", implies that "best" must be interpreted, and this implies not only considerations such as "most direct", or "cheapest", but also "least disruptive to communities". Again the authors assert "…contextuality of all knowledge is (mis)interpreted in a way that leads to the contention that mathematical concepts can be meaningfully learned only within a 'real life' context". Compare this with:

> … neither the content nor vehicle approach argues in some abstract sense that all mathematical curricular content must be justified in terms of relevance - mathematical modelling has a role to play in meeting certain important goals, but other significant mathematical skills and purposes are important as well. (Stillman et.al., 2008, p. 145)

And reasoning that argues against the use of contextualised problems on the grounds that they may be initially more familiar to some students than others should also argue against teaching any new mathematics, because some students will be better prepared to benefit than others. What this paper and others provide, is the cautionary tale that there are many versions of modelling out there, that cover the full range of good, bad, and indifferent implementations. But it is imperative that the theory of mathematical modelling, its purposes and possibilities, are kept conceptually separate from poor implementations, and abuses. There is no question that the latter exist, but they must not be used to undermine arguments for what is possible when the best is undertaken.

Concluding reflection

So, to return to the question posed in the title! It is not reasonable to expect a single definitive answer because not all the 'models' considered have the same priorities. The use of contexts to introduce new mathematical relationships like $y = mx + c$ need to be analysed in terms additional to those raised earlier in this paper. Wrestling with symbolic representations such as m and c, at the same time as embodiments meant to motivate their abstraction creates issues of cognitive load (Chinnappan, 2010) that need to be specifically considered. Emergent Modelling as a package is well constructed by its practitioners, who emphasise and explain what it does and does not set out to do. It is important to realise the curricular implications of its ‘total package’ aspect. Curve fitting will have an increased presence as technology enhances the capacity to use messy real data. It remains a significant *part* of many modelling enterprises, but a challenge is to eliminate any separation between the search for a mathematical relationship, and the nature of the data involved. Word problems will continue their presence, and contain aspects of mathematics congruent with some components of the modelling process. Generally their simplicity and association with text book mathematics, limits their capacity to add decisively to modelling capability.

The last two ‘models of modelling’ share some common ground, and although different in purpose, are not antagonistic. While ‘modelling as a vehicle’ has the prime purpose of eliciting and consolidating *new* mathematical concepts, such entities then enlarge the field of problems that can be addressed. While ‘modelling as real world problem solving’ has the prime purpose of helping students to access and use their *existing* store of mathematical knowledge to address problems, the mathematics evoked is often used in novel ways, and as such contributes to enhanced conceptual understanding. Returning to the aim of the new Australian curriculum, emphasis is on the ability to use mathematics creatively in “personal and work lives and as active citizens”. This requires the ability to formulate mathematical problems out of contextualised settings, and to go through a systematic process of solving, testing, and evaluating. It is not an ability that is acquired by osmosis or transfer on the basis of ‘seeing’—it requires direct structured experience. Formulation of a mathematical problem from a messy real context is arguably the most difficult aspect of learning to use mathematics, and only the two approaches illustrated in this last of the ‘models of modelling’ contain formulation as a major component. It is not surprising that they both resonate with those who apply mathematics outside education.

Finally some comments are needed in response to issues in the previous section raised by Barbosa, Jablonka & Gellert, and Sfard. What each is doing is privileging their conception of what school mathematics is about, and what mathematics teaching and classrooms are allowed to be—then requiring that modelling fit the stereotype and be subject to associated practices. By contrast, what modelling properly conducted can do, is to challenge some of those norms, assumptions, and stereotypes—mathematical, situational, and pedagogical. In that modelling as real world problem solving involves intersections between the values and methods of more than one community of practice, it challenges the boundaries of the existing education industry.

References

Ärlebäck, J. (2009). On the use of realistic Fermi problems for introducing mathematical modeling in school, *The Montana Mathematics Enthusiast, 6*(3), 331–364.

Australian Curriculum, Assessment and Reporting Authority [ACARA] (2010). *Australian Curriculum—Mathematics: Draft consultation* (Version 1.1.0). Retrieved March 15, 2011, from www.australiancurriculum.edu.au/Documents/Mathematics curriculum.pdf

Barbosa, J. (2006). Mathematical Modelling in Classrooms: a socio-critical and discursive perspective. *Zentralblatt für Didaktik der Mathematik, 38*(3), 293–301.

Bardini, C., & Stacey, K. (2006). Students' conceptions of m and c: How to tune a linear function. In J. Novotná, H. Moraová, M. Krátká, & N. Stenhliková (Eds.), *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 113–120). Prague, Czech Republic: PME.

Bardini, C., Pierce, R., & Stacey, K. (2004). Teaching linear functions in context with graphics calculators: Students' responses and the impact of the approach on their use of algebraic symbols. *International Journal of Science & Mathematics Education*, *2*(3), 353–376.

Blum, W., & Leiβ, D. (2007). How do students and teachers deal with modeling problems? In C. Haines, P. Galbraith, W. Blum & S. Khan (Eds.), *Mathematical modelling (ICTMA 12): Education, Engineering and Economics* (pp. 222–231). Chichester: Horwood Publishing.

Blum, W., Galbraith, P., Henn, H-W., & Niss, M. (Eds.). (2007). *Modelling and applications in mathematics education: The 14th ICMI study*. New York: Springer.

Chinnappan, M. (2010). Cognitive load and modelling of an algebra problem. *Mathematics Education Research Journal*, 22(2), 8–23.

Doorman, L., & Gravemeijer, K. (2009). Emergent modeling. Discrete graphs to support the understanding of change and velocity. *Zentralblatt für Didaktik der Mathematik, 41*, 199–211.

English, L. (2010), Young children's early modelling with data. *Mathematics Education Research Journal*, 22(2), 24–47.

Galbraith, P., & Stillman, G. (2006). A framework for identifying student blockages during transitions in the modelling process. *Zentralblatt für Didaktik der Mathematik, 38*(2), 143–162.

Gravemeijer, K. (2007). Emergent modelling as a precursor to mathematical modelling. In W. Blum, P. Galbraith., M. Niss., & H-W. Henn (Eds.), *Modelling and applications in mathematics education: The 14th ICMI study* (pp. 89–98). New York: Springer.

Greer, B., Verschaffel, L., & Mukhopadhyay, S. (2007). Modelling for life: mathematics and children's experience. In W. Blum,, P. Galbraith, M. Niss, & H.-W. Henn (Eds.), *Modelling and applications in mathematics education: The 14th ICMI study.* (pp. 137–144). New York: Springer.

Jablonka, E., & Gellert, U. (2007). Mathematisation—demathematisation. In U. Gellert., .& E. Jablonka (Eds.), *Mathematisation and demathematisation: Social, philosophical and educational ramifications* (pp. 1–18). Rotterdam: Sense Publishers.

Jablonka, E., & Gellert, U. (2011). Equity concerns about Mathematical Modelling. In B.Atweh., M Graven., & W. Secada (Eds.), *Mapping Equity and Quality in Mathematics Education* (Part 2, pp. 223–236). New York: Springer.

Julie, C. (2002). Making relevance relevant in mathematics teacher education. *Proceedings of the second international conference on the teaching of mathematics (at the undergraduate level)* [CD]. Hoboken, NJ: Wiley.

Kaiser, G., & Sriraman, B. (2006). A global survey of international perspectives on modeling in mathematics education. *Zentralblatt für Didaktik der Mathematik, 38*(3), 302–310.

Lesh, R., & Doerr, H. M. (Eds.). (2003). *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching.* Mahwah, NJ: Lawrence Erlbaum Associates.

Pedley, T. J. (2005). Applying mathematics. *Mathematics Today*, 41(3), 79–83.

Riede, A. (2003). Two modelling topics in teacher education and training. In Qi-Xiao Ye., W. Blum., K. Houston., & Qi-Yuan Jiang (Eds.) *Mathematical modelling in education and culture (ICTMA 10)* (pp. 209–222). Chichester: Horwood Publishing.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing.* Cambridge, UK: Cambridge University Press.

Stillman, G., Brown, J., & Galbraith, P. (2008). Research into the teaching and learning of applications and modelling in Australasia. In H. Forgasz, A. Barkatsas, A. Bishop, B. Clarke, S. Keast, W. T. Seah, & P. Sullivan (Eds.), *Research in mathematics education in Australasia 2004–2007* (p. 141–164). Rotterdam, The Netherlands: Sense Publishers.

Stillman, G., Brown, J., & Galbraith, P. (Eds.) (2010). Applications and mathematical modelling in mathematics learning and teaching. Special issue. *Mathematics Education Research Journal 22(2*).

Verschaffel, L., van Dooren, W., Greer, B., & Mukhopadhyay, S. (2010). Reconceptualising word problems as exercises in mathematical modelling. *Journal of Mathematical Didaktics*. 31, 9–29.

Yoon, C., Dreyfus, T., & Thomas, M. (2010). How high is the tramping track? Mathematising and applying in a calculus model-eliciting activity. *Mathematics Education Research Journal*, 22(2), 141–157.

Zbiek, R., & Connor, A. (2006). Beyond motivation: Exploring mathematical modeling as a context for deepening students' understandings of curricular mathematics. *Educational Studies in Mathematics, 63*(1), 89–112.