TWO AVATARS OF TEACHERS' CONTENT KNOWLEDGE OF MATHEMATICS

TRICIA FORRESTER
University of Wollongong
tricia@uow.edu.au

MOHAN CHINNAPPAN
University of Wollongong
mohan@uow.edu.au

In this research we explore pre-service teacher knowledge for teaching mathematics by focusing on the development of the conceptual and procedural knowledge of a cohort of pre-service teachers. In the first phase of this study, we found that a previous cohort of pre-service teachers utilised procedural rather than conceptual knowledge when completing fraction operations. We aimed to address this imbalance by targeting the development of conceptual knowledge through modelling. This paper reports the results of this approach with a subsequent cohort of pre-service teachers, where our expectation of greater conceptual knowledge was achieved and procedural knowledge was maintained.

Introduction

The role of teacher knowledge has been acknowledged as vital in teachers doing their jobs. This issue has been a central concern of the mathematics teaching community both in Australia and elsewhere. We take up this issue in the present study.

Teacher knowledge for teaching mathematics

Shulman's (1986) seminal work on teacher knowledge identified a range of different types of knowledge necessary for teachers to teach effectively. While he acknowledged the essential role of pedagogical knowledge, he highlighted the importance of content knowledge, which he categorised into subject matter knowledge, pedagogical content knowledge and curricular knowledge.

Teachers' mathematical content knowledge affects the quality and nature of their teaching (Schoenfeld, 2000) and has been found to positively predict student achievement (Hill, Rowan, & Ball, 2005). There is little disagreement that teachers need to acquire and understand mathematics in order to teach it effectively.

In acknowledging the multidimensional character of teacher content knowledge, Ball and associates (Ball, Hill, & Bass, 2005; Hill et al., 2005) refined and developed four dimensions of this knowledge: Common Content Knowledge, Specialised Content Knowledge (SCK), Knowledge of Content and Students, and Knowledge of Content and Teaching.

SCK refers to the particular way teachers of mathematics have to understand their content. This involves, among other things, a 'repackaging' of their formal mathematical knowledge. The current study is aimed at better understanding the SCK of prospective teachers. Specifically, we focus on two subsidiary components of SCK, namely, procedural and conceptual knowledge within the domain of fractions. We refer to these strands of knowledge as constituting two *avatars* (Sanskrit word for manifestation) of teacher knowledge in the sense that each of the components are incarnates or embodiments of one key knowledge form, namely, SCK which is the focus on the present study.

Procedural versus conceptual knowledge

Broadly speaking, procedural knowledge involves understanding the rules and routines of mathematics while conceptual knowledge involves an understanding of mathematical relationships. The relationship between procedural and conceptual knowledge, and the dependency of one on the other, continues to be a legitimate concern for mathematics teachers and researchers alike. Schneider and Stern (2010), in examining potential interconnections between the two, suggest that teaching and learning research needs to examine their parallel developments. Within the context of primary mathematics, and in particular fractions, Mack (2001) suggests that children's use of strategies for representing and solving fraction problems are based on both these knowledge strands.

The relationships amongst and the relative role of these two main dimensions of knowledge that is relevant to decoding and solving fractions problems needs further clarification if we are to better inform teachers and knowledge underlying teaching. The debate on this issue appears to proceed along three lines. One view is that children learn conceptual knowledge of fractions before procedural knowledge (Groth & Bergner, 2006). A second view is that children learn procedural knowledge before conceptual knowledge (Baroody, Feil & Johnson, 2007). Finally, it would seem that children's conceptual knowledge and procedural knowledge grow in tandem with one building on the other (Schneider & Stern, 2010). While this debate is continuing, recent research by Hallett, Nunes and Bryant (2010) suggest that a) some children rely on procedural knowledge to inform conceptual knowledge and b) those who rely on conceptual knowledge of fractions tend to have an advantage over those who rely on procedural knowledge. Taken together, these findings suggest that teachers need to have a sound understanding of both these knowledge categories that involve fractions. That is, despite a growing call from some quarters to underplay the role of procedural knowledge in favour of conceptual knowledge (Rittle-Johnson, Siegler, & Alibali, 2001), teachers need to develop a repertoire of both these streams of knowledge as these are legitimate and necessary parts of the corpus of knowledge used by learners that teachers need to know. In this sense conceptual and procedural knowledge are important components of teachers' SCK, and the investigation of this knowledge is a major aim of this study.

Theoretical and Conceptual framework

The aim and analyses of data in the present study are guided by two broad theoretical constructs. In the first instance, we draw on Ball et al.'s (2005) dimensions of teacher knowledge that inform mathematics teaching. Secondly, we examine the interplay between conceptual and procedural knowledge within the *Representational-Reasoning* (RR) model of mathematical understanding provided by Barmby, Harries, Higgins and Suggate (2009). According to this model, the quality of mathematical understanding can be captured by a) the type of representations that learners construct; and b) the robustness of reasoning that is used in establishing or justifying relations among the representations. We see the RR model as somewhat unbiased in the interpretation of the relative roles of conceptual and procedural knowledge, as both components can be foregrounded in the representations and reasoning.

Issues and aim

The review of literature on teacher knowledge and teachers' performance in relation to children's numeracy levels has highlighted the need to research and monitor the developing knowledge of mathematics teachers who are in practice and those who are in training. Initially we investigated this issue by analysing the procedural and conceptual knowledge of fraction operations of a cohort of pre-service teachers (Forrester & Chinnappan, 2010). The results of this analysis demonstrated clearly the dominance of procedural knowledge over conceptual knowledge in this group, with almost four times the number of pre-service teachers activating procedural knowledge in comparison to those that demonstrated conceptual knowledge in their solution attempts. About one fifth of responses evidenced neither procedural nor conceptual knowledge.

While both knowledge categories are important, the dominance of one over the other would seem to be unhealthy for classroom practice, as teachers will have to support the development of both procedural and conceptual knowledge in their students across all strands of primary mathematics including fractions. This line of reasoning motivated us to modify our teaching strategies with the view to enhancing the conceptual component of our pre-service teachers' knowledge of fractions.

The aim of this study was, therefore, to ascertain the impact of a model based teaching (MBT) approach on the development of procedural and conceptual knowledge in the domain of fractions. This guided us in the development of the following research questions:

1. Does a model-based teaching approach have an impact on the development of pre-service primary teachers' procedural knowledge of fractions?
2. Does a model-based teaching approach have an impact on the development of pre-service primary teachers' conceptual knowledge of fractions?

Methodology

Participants

Two hundred and twenty-four students (37 males and 187 females) participated in the present study. They were enrolled in a first year compulsory subject, which is generally completed in the second semester of a four-year Bachelor of Primary Education degree. Prior to entry into the program, the participants had a range of mathematical backgrounds.

Procedure

Model-based teaching

Subsequent to the analysis of the 2009 cohort of pre-service teachers' conceptual and procedural knowledge of fraction operations discussed earlier (Forrester & Chinnappan, 2010) changes were made to the delivery of the subject in 2010. Utilising Barmby et al.'s (2009) notion that robust mathematical understanding is demonstrated when learners can construct and utilise multiple representations of mathematical ideas and can justify the relationships among representations, we focused on enabling our students to develop models of fraction operations and appropriate explanations of these models (MBT approach).

Tasks

The following tasks were two parts of one question in a fifteen-question examination. They were selected from a pool of thirty-five questions given to students in their subject outlines at the beginning of the semester. These particular tasks were chosen to examine students' mathematics content knowledge in terms of their conceptual and procedural knowledge of fractions and fraction algorithms. While the fractions were different from those given in the subject outline, the format of the questions was identical and students had been able to engage with similar questions throughout the session to consolidate their procedural and conceptual understandings.

Task 1: Division problem involving a mixed number and fractions with different denominators

$$1\tfrac{1}{2} \div \tfrac{1}{4}$$

Two algorithms could be used to complete this task. Firstly students could: change the mixed number into an improper fraction; invert the divisor; multiply the numerators and denominators; check if the answer can be simplified. Alternately students could: change the mixed number into an improper fraction; identify a common denominator of the dividend and divisor; change the dividend and divisor to equivalent fractions; divide the numerators and denominators; check if the answer can be simplified.

One conceptual understanding of this task involves the notion that $1\tfrac{1}{2} \div \tfrac{1}{4}$ involves finding how many $\tfrac{1}{4}$s are in $1\tfrac{1}{2}$. Partitioning $1\tfrac{1}{2}$ into quarters and counting the number of quarters will achieve an answer of 6.

Task 2: Addition problem involving a mixed number and fractions with different denominators

$$1\tfrac{5}{6} + \tfrac{2}{3}$$

Again, two algorithms could be used to complete this task, both involve most or all of these procedures: changing the mixed number to an improper fraction; identifying a common denominator of the addends; changing the addends to equivalent fractions; performing the addition; checking if the answer can be simplified. A conceptual knowledge of this task involves these elements: when the addends are modelled visually the *wholes* to which they relate are the same size; equivalent fractions e.g., $1\tfrac{5}{6}$ is the same size as $\tfrac{11}{6}$, $\tfrac{2}{3}$ is the same size as $\tfrac{4}{6}$, $\tfrac{15}{6}$ is the same size as $2\tfrac{3}{6}$ which is the same size as $2\tfrac{1}{2}$; addition involves joining two or more quantities together.

In undertaking the tasks, students were asked to complete the calculations and provide models and explanations of their models. They needed to use an appropriate algorithm for carrying out the required operation with fractions. The successful use of an appropriate algorithm would indicate that students have a procedural understanding and concomitant use of procedural knowledge. Conceptual understanding of these tasks involves demonstrating the nature of fractions (equal parts of a whole object or group) including the meaning of the common fraction symbol—as opposed to the misconception common among children that the numerator and denominator are simply two whole numbers (NSW Department of Education and Training, 2003). Additionally, a conceptual understanding of the tasks involves grasping what happens when dividing and adding fractions, including the relationship between the fractions involved.

Coding scheme

Students' responses to each of the two problems were analysed in terms of their demonstration of conceptual and procedural knowledge, and coded using a ten code scale (see Table 1). This coding scheme is a refinement of the one used to analyse the data collected and analysed previously (Forrester & Chinnappan, 2010) which was developed using the theoretical framework of Barmby et al. (2009) and Goldin's (2008) analysis of problem representations. We wanted to modify our codes to allow for greater differentiation of responses, in terms of conceptual and procedural knowledge. This scale also includes a code for a category of responses that did not occur in the previous data (Code 8), where a correct solution was achieved through a conceptual model and no algorithm was utilised.

Table 1 - Coding Scale.

Code	Algorithm	Model/Explanation
0	None provided	None provided
	Explanation: Where there is a response it is just an answer with no algorithm or model/explanation.	
1	Inappropriate	None or incorrect conceptual representations
	Explanation: An algorithm has been used but it is not appropriate for the problem. If a model/explanation has been provided it is incorrect conceptually.	
2	Inappropriate with correct elements	None or incorrect conceptual representations
	Explanation: While the algorithm used was inappropriate to the problem, important fraction processes were used e.g., making equivalent fractions, changing mixed numbers to improper fractions. If a model/explanation was provided it was incorrect conceptually.	
3	Appropriate but errors made	None or incorrect conceptual representations
	Explanation: The algorithm used was appropriate for the problem but an error occurred in its use. If a model/explanation was provided it was incorrect conceptually.	
4	Appropriate, used correctly	None or incorrect conceptual representations
	Explanation: The algorithm used was appropriate for the problem and achieved a correct answer. If a model/explanation was provided it was incorrect conceptually.	
5	Appropriate but errors made	Some level of conceptual representation
	Explanation: The algorithm used was appropriate for the problem but an error occurred in its use. The model/explanation utilises some level of conceptual representations.	
6	Appropriate, used correctly	Thorough procedural representation No conceptual representation
	Explanation: The algorithm used was appropriate for the problem and used to achieve a correct answer. The model/explanation was a detailed representation of the algorithm but did not demonstrate the concepts involved in fraction operations.	
7	Appropriate, used correctly	Some level of conceptual representation
	Explanation: The algorithm used was appropriate for the problem and used to achieve a correct answer. The model/explanation utilises some level of conceptual representation.	
8	None provided	Strong conceptual representations.
	Explanation: No algorithm was used. A correct answer was achieved using a conceptual model.	
9	Appropriate, used correctly	Strong conceptual representation
	Explanation: The algorithm used was appropriate for the problem and used to achieve a correct answer. The model/explanation was conceptually correct.	

Inter-rater reliability analysis

In order to determine the reliability of the coding scheme, we assessed the extent to which two coders agreed when they independently categorised students' responses. The two researchers coded twenty-five students' responses independently. The inter-coder reliability analysis, using the Kappa statistic, was performed to determine consistency. The inter-coder reliability was found to be Kappa = 0.77 ($p < 0.001$), 95% CI (0.504, 0.848), indicating substantive agreement (Landis & Koch, 1977) in the way the students' responses were coded by each researcher. Potential areas of disagreement were analysed which helped us to improve the distance between the codes, thereby reducing areas of ambiguity.

Data and analysis

Quantitative data analyses were conducted with the aid of SPSS version 18. Our analyses focused on the above ten categories of problem representation; the scale of our data was nominal.

In this paper we report the results of our analysis of student examination responses following a semester of lectures and tutorials that focused on developing conceptual and procedural knowledge through modelling and explanation (2010 cohort). We compare these results with those reported previously (2009 cohort) (Forrester & Chinnappan, 2010).

The data were analysed in terms of the two research questions:

1. *Does a model-based teaching approach have an impact on the development of pre-service primary teachers' procedural knowledge of fractions?*

The proportion of pre-service teachers who were able to find correct solutions to the fraction operation tasks using algorithms was not considerably different over the two years. Of the 2010 cohort, 74.6% (÷) and 76.4% (+) of participants were able to demonstrate the competent use of appropriate algorithms to achieve correct solutions in the division and addition tasks respectively (See Figures 1 and 2 - Codes of 4, 6, 7, 9). Within the 2009 cohort, 79.6% (x) and 72.6% (−) of participants were able to demonstrate the competent use of appropriate algorithms to achieve correct solutions in the multiplication and subtraction tasks respectively.

The proportion of pre-service teachers unable to achieve a correct answer using procedural or conceptual knowledge decreased slightly over the two years: Of the 2009 cohort, 20.4% (x) and 27.4% (−) of participants did not achieve correct solutions in the multiplication and subtraction tasks respectively. Within the 2010 cohort, 17.9% (÷) and 22.9% (+) of participants did not achieve correct solutions within the division and addition tasks respectively (See Figures 1 and 2 - Codes 0, 1, 2, 3, 5).

The impact of model-based teaching on pre-service teachers' procedural knowledge is somewhat unclear because the data collected in 2009 and 2010 are not markedly different.

2. *Does a model-based teaching approach have an impact on the development of pre-service primary teachers' conceptual knowledge of fractions?*

The majority of pre-service teachers in the 2010 cohort exhibited strong conceptual understanding, with 65.6% (÷) and 55.8% (+) (See Figures 1 and 2 - Codes of 8 and 9) being able to successfully model and explain the mathematical concepts involved in division and addition operations. A further 1.8% (÷) and 4.9% (+) were able to demonstrate some level of conceptual understanding (See Figures 1 and 2 - Code of 7).

Interestingly, 7.6% of participants (17 students) were able to achieve a correct answer to the division task (÷) utilising a conceptual model without utilising an algorithm. Two participants (0.9%) achieved a correct solution to the addition task (+) utilising a conceptual model without using an algorithm (See Figures 1 and 2 - Code of 8). Given that students were required to provide a calculation, any omission in providing evidence of procedural knowledge can be interpreted as not having this knowledge.

In comparing these results with those of the previous cohort (2009), it seems that a model-based approach to teaching has contributed to the substantial differences in our pre-service teachers' demonstration of conceptual understanding of fraction operations over the period of this research. There are difficulties in making direct comparisons between these sets of data as we examined multiplication (x) and subtraction (−) in 2009 and division (÷) and addition (+) in 2010. However, many of the concepts and procedures in finding a solution for these tasks are the same. In 2009, 11.8% of participants could demonstrate conceptual knowledge in multiplication (x) and 18.8% in subtraction (−). In the present study, 65.6% of participants evidenced conceptual knowledge in division (÷) (58% demonstrating both conceptual and procedural knowledge) while 55.8% could demonstrate conceptual knowledge in addition (+) (54.9% demonstrating both conceptual and procedural knowledge). We regard this as supporting our expectation of the positive impact of the model-based approach.

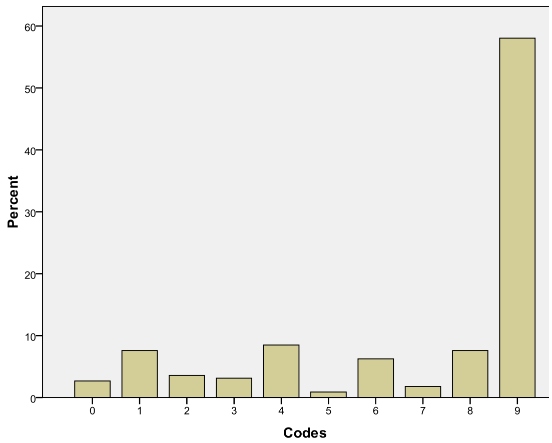

Figure 1: Coding frequency for Division.

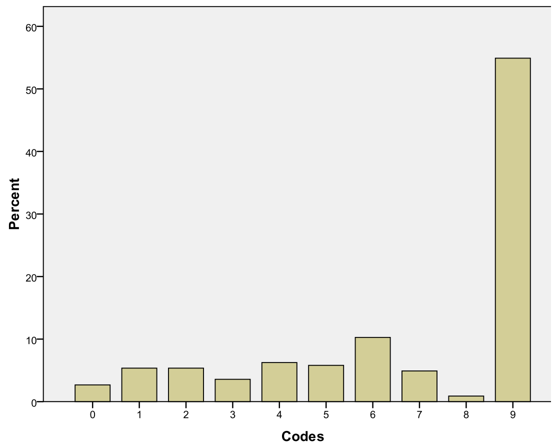

Figure 2: Coding frequency for Addition.

Discussion and implications

The study was grounded on the assumption that teachers' Specialised Content Knowledge of Mathematics (Ball et al., 2005) needed to have both procedural and conceptual characteristics in the domain of fractions. While conceptual knowledge may subsume procedural knowledge and indeed contribute to a better understanding of related procedural knowledge, it is important to capture and support both strands of knowledge for future teachers of mathematics.

The research questions were concerned with the impact a teaching approach that was based on modelling would have on the development of pre-service teachers' procedural and conceptual knowledge of fractions. The results here suggest that, while there was no tangible effect on procedural knowledge, our teaching had a positive effect on pre-service teachers' conceptual knowledge. The design of the MBT was guided, in the first instance, by an analysis of the state of pre-service teachers' knowledge within a narrow domain of context-free fraction problems. This analysis, it would seem, is critical for the design of MBT for fractions or similar approaches for other areas of primary mathematics. MBT was also framed around the notions of representations and reasons (Barmby et al., 2009) which aided us in visualising the role of conceptual and procedural knowledge in comprehending and making progress with the fraction problems.

The role of Barmby et al.'s (2009) framework in the development of MBT constitutes an important outcome of this research. We found the framework to be useful in drawing the distinction between procedural and conceptual knowledge, and how these two strands of knowledge interact and constrain the construction of representations.

The MBT approach was based on the assumption that pre-service teachers who had developed robust conceptual knowledge could be expected to exhibit strong procedural knowledge. This appears to be the case with most of our participants. However, there were a number of pre-service teachers who demonstrated conceptual knowledge but failed to activate the corresponding procedural knowledge. This raises a question about the character of conceptual knowledge in subsuming and supporting procedural knowledge. This issue needs further analysis and the subject of future investigations.

The results showed that pre-service teachers have developed strong conceptual understanding of division problems. However, the robustness of this understanding needs to be the subject of further research including the analysis of prospective teachers' representations and solutions of division problems that are contextualised. The representation of division problems, both from a conceptual and procedural point of view, could inform us about pre-service teachers' ability to discriminate measurement versus partitive interpretations which have been shown to be a problematic area for teachers and students (Flores, 2002; Siebert, 2002).

Is conceptual knowledge better than procedural knowledge for practice? We suggest that there has to be a balance and that teachers' SCK ought to exhibit both these characteristics. Equally, teachers need to be facile in articulating their relationships.

Our previous study (Forrester & Chinnappan, 2010) provided the impetus for this project. In that study we examined procedural and conceptual knowledge in the context of subtraction and multiplication problems. The current study, however, involved the investigation of addition and division problems. This could be seen as a limitation. We contend that in both situations, there is a structural similarity (inverse relationships) among these operations, both algorithmically and conceptually.

References

Ball, D. L., Hill, H. C., & Bass, H. (2005). Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator, 5*(3), 43-46.

Barmby, P., Harries, T., Higgins, S., & Suggate, J. (2009). The array representation and primary children's understanding and reasoning in multiplication. *Educational Studies in Mathematics, 70,* 217-241.

Baroody, A. J., Feil, Y. & Johnson, A. R. (2007). An alternative reconceptualisation of procedural and conceptual knowledge. *Journal for Research in Mathematics Education*, 38**,** 115–131.

Flores, A. (2002). Profound understanding of division of fractions. In B. Litwiller & G. Bright (Eds.), *NCTM Handbook: Making sense of fractions, ratios and proportions* (pp. 237-246). Reston, VA: National Council of Teachers of Mathematics.

Forrester, P. A., & Chinnappan, M. (2010). The predominance of procedural knowledge in fractions. In L. Sparrow, B. Kissane, & C. Hurst (Eds.) (2010). *Shaping the future of mathematics education* (Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia, pp. 185-192). Fremantle, WA: MERGA.

Goldin, G. A. (2008). Perspectives on representation in mathematical learning and problem solving. In L. English (Ed.), *Handbook of international research in mathematics education*. New York: Routledge.

Groth, R. E. & Bergner, J. A. (2006). Preservice elementary teachers' conceptual and procedural knowledge of mean, median, and mode. *Mathematical Thinking and Learning,* 8**,** 37–63.

Hallett, D., Nunes, T., & Bryant, P. (2010). Individual differences in conceptual and procedural knowledge when learning fractions. *Journal of educational psychology, 102*(2), 395-406.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of Teachers' Mathematical Knowledge for Teaching on Student Achievement [Electronic version]. *American Educational Research Journal, 42*(2), 371-406.

Landis, J. R., & Koch, G. G. (1977). The measurement of observer agreement for categorical data. *Biometrics*, *33*, 159-174.

Mack, N. K. (2001). Building on informal knowledge through instruction in a complex content domain: Partitioning, units, and understanding multiplication of fractions. *Journal for Research in Mathematics Education*, *32*(3), 267-295.

NSW Department of Education and Training. (2003). *Fractions, pikelets and lamingtons*. Ryde, NSW: Author.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An interative process. *Journal of Educational Psychology, 93*(2), 346-362.

Schneider, M., & Stern, E. (2010). The developmental relations between conceptual and procedural knowledge: A multimethod approach. *Developmental Psychology, 46*(1), 178-192.

Schoenfeld, A. H. (2000). Models of the Teaching Process. *Journal of Mathematical Behavior, 18*(3), 243-261.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 4*(15), 4-14.

Siebert, D. (2002). Connecting informal thinking and algorithms: The case of division of fractions. In B. Litwiller & G. Bright (Eds.), *NCTM Yearbook: Making sense of fractions, ratios and proportions* (pp. 247-256). Reston, VA: National Council of Teachers of Mathematics.