TEACHERS' STRATEGIES FOR DEMONSTRATING FRACTION AND PERCENTAGE EQUIVALENCE

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Understanding the relationships among fractions, decimals, and percentages is a critical goal of the middle years of schooling. There are many approaches that teachers might take to help students develop this understanding; some capture general principles whereas others only illustrate specific equivalences. In this study teachers were asked to suggest three ways of convincing students that three-eighths is the same as 37.5%. The data reveal a wide range of strategies and show that different approaches may exemplify different features of the fraction-percentage relationship. The explanatory power of the examples is also considered.

Background

The development of thinking about fractional quantities is one of the notorious mountains in school mathematics (e.g., Behr, Harel, Post, & Lesh, 1992; Litwiller & Bright, 2002). Even before worrying about computation with fractional quantities, students need to understand the meaning of these quantities as numbers (Kilpatrick, Swafford, & Findell, p. 235). It is well known that students have difficulty identifying the whole, coordinating the values of the numerator and denominator, and being able to treat the fraction as a single number that can then be related to other quantities. The work of Clarke and Roche (2009) highlighted the difficulties that students have with fraction comparisons, associated with the problems identified above, and pointed out the effectiveness of having knowledge of benchmark fractions such as 1/2 as a point of reference.

Added to these challenges is the fact that fractional quantities can be represented in at least four distinctive ways: as rational numbers in fraction form, as ratios, as decimals, and as percents. Although the last two are closely linked—to the extent that the digits are identical in their representations—it is not always easy for students to appreciate what the % symbol signifies, and that what appears to be a whole number (or, at least, a number greater than 1), is actually a fractional part of 1. What is more, there are many different concrete illustrations of these representations. Students may encounter fractions being illustrated using area models, fraction strips (or fraction walls), number lines, and set models, with each model perhaps highlighting certain aspects of fractions. Sowder (1988) observed that many children are "model poor", having only a circular model to represent fractional quantities. This limited representation restricts access to some of the relationships that are important to establish, not least of which is the connection to decimals and percents. The idea of *epistemic fidelity* is useful here (see, e.g., Stacey, Helme, Archer, & Condon, 2001), since it highlights the need for models and representations to accurately capture the mathematical features of the concept they are trying to represent.

Knowing about and using suitable representations in teaching has long been recognised as a significant component of pedagogical content knowledge (PCK). In his seminal paper, Shulman (1986) wrote:

> [Pedagogical content knowledge includes knowledge of] the most useful form of representations of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others. … The teacher must have at hand a veritable armamentarium of alternative forms of representation … . (p. 9)

Thompson and Thompson (1996) also highlighted that teachers need conceptual schemes that incorporate a clear picture of the materials, activities, and explanations that will facilitate the development of mathematical understanding in students. Other researchers (e.g., Askew, Brown, Rhodes, Johnson, & Wiliam, 1997; Ma, 1999) have pointed to the significance of teachers being able to make connections among and within topics in order to improve students' learning outcomes.

While brief, the discussion above highlights that fraction teaching has the potential to incorporate a wide range of models or representations, although Sowder (1988) suggests that students use few of them. Through these representations it may be possible to help students develop connected understanding of fractions, decimals, and percents. As Askew et al. (1997) suggest, however, this will depend on the way teachers use the representation or model, and what connections they can make explicit with it. The purpose of the present study, therefore, was to examine what representations teachers draw on when talking about fractions and percentages, and what strategies they have for helping students to develop appropriate connections.

Method

Participants

Four cohorts of teachers—pre-service and practising—provided data for this study. These cohorts are described below, but when referring to all the participants as a single group the term "teachers" will be used, irrespective of whether or not they were employed in that role. Note that the data from the pre-service teachers were initially analysed in Chick (2003), but have been reanalysed for this paper.

Pre-service teachers

DipEd cohort

Participants from the DipEd cohort (N=16) were preparing to become secondary mathematics teachers. They had studied tertiary-level mathematics to at least sub-major level, prior to undertaking a one-year Diploma of Education. The mathematics method unit that they were doing at the time of the study focused on the teaching of secondary level mathematics topics.

BEd cohort

Those in the BEd cohort (N=21) were in the final year of a four-year Bachelor of Education degree preparing to become primary teachers, and prior to this had completed mathematics to at least Year 11. The mathematics units in the BEd program covered elementary mathematics content and pedagogy simultaneously, and at the time of the study the cohort had completed nearly six semesters of such units. The content was mainly concerned with primary school level mathematics, and had included fractions and decimals. The timing of the study, and the fact that participation was voluntary, makes it likely that these pre-service teachers were the more mathematically confident of all the BEd students that year, and so the results for the BEd cohort may be inflated.

Practising teachers

Secondary cohort

Participants from the secondary cohort (N=40) came from three Victorian metropolitan government secondary schools. They comprised all those members of staff in the schools involved in teaching one or more classes of mathematics. As a result the secondary cohort included specialist mathematics teachers, mathematics and science teachers, and some teachers teaching outside their area of formal qualification. Their teaching experience ranged from being in their first year through to more than 20 years in the classroom.

Primary cohort

The practising primary teachers (N=15) were Grade 5 or 6 teachers from a variety of government, Catholic, and independent schools around Victoria. Their paths to their teacher qualification were diverse: some had a four-year education degree, others had a degree in another discipline followed by a one-year education diploma. The range of the number of years of teaching was as diverse as the secondary cohort. These teachers were volunteer participants in a study on PCK in mathematics, and may have a greater degree of mathematics teaching confidence than primary school teachers in general.

The task

All cohorts responded to a written questionnaire addressing a wide range of topics associated with pedagogical content knowledge for mathematics. The questionnaires varied for the different cohorts, but the following item was common to all of them: "Write down three ways of convincing someone that 3/8 is the same as 37.5%". This item is the focus of the present study. Space was provided on the questionnaire for three written suggestions. The practising teachers were also interviewed about their responses to the questionnaire items, but for the purposes of this paper only the written responses were analysed. There was one exception, a teacher who provided interview data only.

Analysis

As the data were analysed and entered into a spreadsheet, each different method used in a response was given a code to indicate the way in which the relationship between 3/8 and 37.5% was demonstrated (e.g., by doing a division algorithm with 3÷8, by using an area model, by using a dual number line, and so on). Additional annotations were made to record any variations from the standard response types. A total of 32 different methods were observed, although some of these were similar and so the data could, perhaps, have been condensed into fewer categories. In the case where one of the methods suggested by a teacher was essentially equivalent to one of his/her earlier suggestions this was noted as a "repeat" in order to identify distinct strategies.

In addition to having the nature of the method recorded, each method was rated "good", "okay", or "incorrect/inadequate", depending on its adequacy as a "way of showing someone that 3/8 is the same as 37.5%". Although it could be argued that applying the algorithm 3/8 x 100/1 is not a convincing demonstration of equivalence, it is a standard approach to determining the relationship between a fraction and its corresponding percentage value, and so was rated as "good". Indeed, it is the only strategy that is readily applicable in some circumstances, such as with awkward fractions. "Use a calculator" was also rated as "good", on the grounds that, provided the operations chosen are deemed appropriate, the calculator has a sort of computational authority. Other explanations or demonstrations that were clear and had the power to convince were also rated "good". Methods that rated "okay" were those that were partially correct/convincing, incomplete, or difficult to implement. Examples include asserting 3/8 = 0.375 without justification, trying to show 37.5% by dividing a "pie chart" into 100 wedges, or writing "turn 37.5% into a fraction" without discussing how. Finally, the rating of "incorrect/inadequate" was given to responses which were not clear or which were erroneous or lacking key details (such as "guess and estimate").

The data for the present study were initially coded and rated by the second author, and then checked by the first author.

Results and discussion

The results are presented in two parts. First, an overall picture of the number and quality of the suggestions made by the teachers will be provided. This will give an indication of the number and appropriateness of the representations and explanations that teachers had at their disposal for dealing with fractions and percentages. The second subsection reports more specifically on the different kinds of techniques that were proposed, and examines the "explanatory power" of some of the methods.

Number and quality of responses

The number and quality of the responses, for each of the four cohorts, is shown in Table 1, which gives a detailed breakdown of the distribution of "good", "okay" and "incorrect/inadequate" responses. Overall, 82% of the teachers (DipEd 81%, BEd 71%, Secondary 83%, Primary 93%) were able to come up with at least one "good" response, although this may simply have been to apply the fraction-to-decimal computation or to "use a calculator". In contrast, only 38% were able to come up with three distinct "good" or "okay" methods (DipEd 44%, BEd 14%, Secondary 50%, Primary 40%), and fewer than 20% could provide three distinct "good" methods (see line 1 of Table 1). All of the practising teachers were able to suggest at least one "okay" method or better.

Looking at the cohorts together, nearly one-third could not provide what the participant adjudged to be three suitable methods (regardless of whether they were rated by the researchers as suitable or not, or a repeat). The pre-service teachers, in particular, struggled in this area, with over half of them failing to find three methods, and a handful failing to propose any methods. This suggests that experience and professional development *do* provide opportunities for growth in expertise. Having said this, however, it is of concern that a quarter of all the teachers—with this proportion applying to the practising teachers as well—made at least one suggestion that was wrong or seriously inadequate.

Table 1. Percentage of teachers and the number of appropriate methods proposed.

Number of methods provided and their value*	DipEd (N=16)	BEd (N=21)	Secondary (N=40)	Primary (N=15)
Provided 3 or more distinct “good” methods	25%	14%	18%	20%
Provided 2 “good” and 1 “okay” distinct methods	13%	0%	28%	7%
Provided 2 “good” and 0 “okay” distinct methods	13%	14%	20%	20%
Provided 1 “good” and 2 or more “okay” distinct methods	6%	0%	3%	13%
Provided 1 “good” and 1 “okay” distinct methods	13%	33%	10%	33%
Provided 1 “good” and 0 “okay” distinct methods	13%	10%	5%	0%
Provided 0 “good” and at least one “okay” distinct methods	6%	10%	18%	7%
Provided no “good” or “okay” suggestions	13%	19%	0%	0%
Unable to get 3 methods (regardless of correctness or repetition)	44%	57%	18%	27%
Did not provide any suggestions	6%	14%	0%	0%
Provided at least one “incorrect/inadequate” suggestion	13%	38%	25%	27%
Provided 4 “good” or “okay” distinct suggestions	0%	0%	8%	0%
Provided a 4th suggestion (may not have been “okay”, and nor may the earlier ones have been)	6%	0%	8%	7%
Number of repeated/equivalent suggestions	6%	5%	13%	20%

Methods for showing the equivalence of 3/8 and 37.5%

In all, the 92 teachers provided 236 methods that they felt were appropriate for showing the equivalence of 3/8 and 37.5%. As mentioned earlier, 32 codes were used to identify the different methods or strategies, but there were some commonalities that allow the methods to be grouped loosely. These categories are described below, and their distributions are indicated in Table 2. In some cases the strategies suggested by the teachers were not described completely with necessary connections made explicit (so that a reader could not be certain that the explanation would be implemented successfully), but if the underlying principle was evident it was grouped into the appropriate category even if it had been rated as “incorrect/inadequate”.

Computational approaches

As can be seen in Table 2, the most common approaches were computational (42% of all suggestions), with the prevalent strategy among these to compute 3/8 x 100/1 (suggested in 17% of the responses overall), which the teachers usually did by hand. This approach, like most of those placed in this category, has little explanatory power: the person to be convinced about the relationship has to accept that the computation does, indeed, convert a fraction to a percentage. Other strategies included in this category were using a calculator (proposed in 11% of the suggestions, usually without explaining what operations were necessary), applying a division algorithm to 3÷8 or 300÷8, and converting both 3/8 and 37.5% to decimals. To be placed in this category there had to be a sense of the formulaic application of an algorithm or calculating without deep attention to relationships. This is not meant to devalue the approach, but to highlight what may or may not be conveyed by it. As noted earlier, there are some fraction to percentage conversions that will *only* be possible by such a method, since some of the strategies in the remaining categories below will not work so readily for things like “convert 7/9 to a percentage”.

Numerical relationships

Many of the explanations took advantage of the numerical relationships among the quantities, and used these relationships to establish the result. One of the most common of these approaches was to establish a sequence of fractions equivalent to 3/8, from which 37.5% could be obtained (for example, 3/8 = 75/200 = 37.5/100). An alternative was to start from 37.5% and establish the result via 37.5/100 = 375/1000 and then cancel common factors. About 11% of all responses used one or other of these strategies. Still others wrote that “3/8 = something/100” and then used algebra or equivalent fractions to establish the value of “something”. The numerical relationships in all of the above methods were, in general, readily established by mental computation. As was the case for the algorithmic/computational approaches, in most cases there were implicit assumptions about the meaning of percentage: the equivalence of 37.5% and 37.5/100 was assumed without explanation. The other family of responses grouped with this category of “Numerical relationships” used benchmark values to establish equivalences, such as working from 50%=1/2 (assumed to be well-known) to establish 25%=1/4 and 12.5%=1/8, and thus 3/8=37.5%. About 5% of all the responses used this benchmark approach.

Diagrammatic representations

About 18% of the responses proposed some diagrammatic or visual representation to establish the relationship. Most of these involved area models but of the 14% of responses that attempted such a representation fewer than half were convincing. Most of the successful ones established eighths in a region (often a circle), and obtained the relationship of 1/8=12.5% from the assumed to be well-known relationship between 50% and 1/2 (see also the discussion about numerical relationships above). One unusual successful example involved a 10x10 grid, in which every eight squares were identified and three of these coloured in. The teacher’s description successfully dealt with the four squares remaining at the end. The more problematic examples included (a) attempting to represent 37.5% in a “pie cut into 100” without considering whether this could be done in practice, let alone showing how this is actually the same as 3/8 of the same circle, (b) showing 3/8 of a circle and asserting its equivalence with a square 10x10 grid shaded to show 37.5%, and (b) suggesting “cutting cake” with no further detail. Other diagrammatic approaches used the circumference of a circle rather than the area (not done successfully), or used number lines, with only two teachers proposing an appropriate dual number line. One particularly nice representation used a 1 m measuring tape and folded it into eighths, and then measured the length of 3/8.

Asserted results

In a number of responses (7% overall) the teachers had an appropriate explanation or sequence of computational steps, with the exception of an unexplained jump from 3/8 to 0.375 (or, as was done by some of the teachers, from 1/8 to 0.125). Responses were put in this category if this equality was asserted without explanation. It may be that the teachers envisaged demonstrating the equality on a calculator or by some other means; or it may have been a known fact for them but they may not have realised that they were also assuming that it was known to the recipient of their explanation. Establishing this equality seems to be part of the requirement of the explanation, along with developing the more general connection between a fraction and its associated percentage.

Use of meanings

A small number of responses (3%) made explicit use of the meaning of division in trying to establish the equivalence of 3/8 and 37.5%. For example, two teachers suggested taking 100 objects, sharing them among 8 groups, and seeing how many objects are in 3 groups. The other interesting approach interpreted both the fraction and the percentage as operators and suggested calculating both 3/8 of some number and 0.375 of the same number.

Unclear or tautology

This category was reserved for those strategies in which it was not clear what the teacher intended to do or how, or where the teacher made a tautological assertion. As examples of the former, teachers wrote "Estimation" (secondary), "Compare with a fraction like 4/8 = 1/2" (secondary), "Measuring volume of water" (primary), "Make them divide to two decimal points [sic]" (DipEd), and "Using a protractor" (BEd). As examples of tautological assertions, one teacher wrote "30/80 = 3/8 = 37.5/100" (secondary), with no indication of how these relationships—notably the final one—were established, whereas another wrote

> If you completed a test that was out of 100 and if you received a grade [of] 37.5 which is the same as 37.5%. What if the test was out of 8 instead of 100. 37.5 out of 100 is the same as 3 out of 8. (Primary)

Three of the teachers wrote that the 3/8 = 37.5% relationship holds "because it is" or suggested, "tell them [students] to trust you because you are the teacher".

Table 2. Percentage of methods by type.

Method type	DipEd (N=36*)	BEd (N=43*)	Secondary (N=115*)	Primary (N=42*)
Computational (limited "demonstrative" power)	47%	42%	42%	41%
Uses numerical relationships	33%	12%	22%	21%
Diagrammatic representations (model or illustration)	8%	21%	19%	19%
Asserted a non-obvious result without explanation	6%	5%	9%	5%
Uses meaning of 3/8 or 37.5	0%	5%	3%	5%
Unclear or "it is"	6%	16%	6%	10%

* Here N is the total number of methods proposed by the cohort.

Conclusions

Examining the results across the cohorts, it appears that the pre-service BEd students did not have as many successful strategies at their disposal as their practising and secondary-oriented counterparts. Nevertheless, they suggested appropriate diagrammatic strategies in the same proportion as the practising teachers, and better than the DipEd cohort, perhaps because the latter cohort had reliable personal computational skills at the same time as having had limited opportunities to develop or learn other strategies for assisting students.

Although a wide range of strategies was presented, the prevalence of routine computation was striking. The frequent use of circle models for area reinforces Sowder's 1988 finding of their abundant use, and so perhaps we now know why Sowder's students used circle models almost exclusively: they learned from their teachers.

Finally, some further thought needs to be given to what each of the different methods make transparent and what is obscured. The computational approach works for every possible fraction yet it appears to hide the fundamental relationship between a fraction and its decimal value in a computation that it is possible to conduct—and teach about—almost mindlessly. On the other hand, the use of "nice" relationships and convenient benchmarks to determine the equivalence of 3/8 and 37.5% would only be generalisable to a few special cases. At the same time, however, such methods build facility with mental computation, allow work with benchmarks, and are perhaps better able to indicate the underlying connections between fractions and percentages. Furthermore, these "special" cases may be useful for motivating and justifying the more algorithmic approaches, and for highlighting why they are necessary. The significance of the teaching opportunities that become available because of access to multiple representations cannot be understated.

There are clear implications for teacher preparation and professional learning. Not only is it important that teachers have access to multiple representations that give students alternative models for concepts, but teachers need to realise that some models are better than others for highlighting particular mathematical features. What one model makes obvious may be obscured or difficult to see in another model. Teachers also need to be aware about the generalisability of a representation and of the importance of special cases that can be used to illustrate more general connections that might be difficult to make clear using more arbitrary examples.

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