# Do Algebraic Skills Improve with Higher Learning?

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Many students enrolled in first year university mathematics displayed consistent misunderstanding with some essential algebraic skills. Analysis of an algebraic test given to first year university students in 1996 highlighted five major categories of concern. Further algebraic tests were given to 1997 first year university mathematics students, second year university mathematics students and senior secondary students. The five categories were confirmed as major areas of consistent algebraic difficulties. Many of these difficulties did not seem to improve with higher mathematical learning.

## Introduction

Understanding is important for success with algebraic skills. Many students know variables only as placeholders for numbers and are totally unprepared to regard algebraic entities in different ways (Barbeau, 1995).

How can we expect students to interpret solutions to problems when basic algebraic difficulties, based on a lack of understanding of early concepts that are assumed to be mastered prior to university, can be their largest stumbling block?

In 1996, two groups of first year university students (N=526 and N=88) sat an algebraic test three weeks after enrolment in a general mathematics course. Detailed analysis of these scripts led to the detection of five topic-independent categories of incorrect responses (Boustead, 1996). Four of these categories were described by Rotman as exclusive pre-algebra areas of arithmetic (Rotman, 1991). Rotman labelled the categories *"understanding the order of operations agreement"*, *"understanding the properties of numbers, especially fractions"*, *"understanding the structure of obtaining solutions"* and *"understanding the meaning of symbols"*. The last category was relabelled by this author as *"false generalisations"*. A final category of incorrect responses added after the analysis of the 1996 algebraic test was labelled *"understanding the range of possible solutions"*. Three of these categories were also mentioned in an analysis of a multi-choice algebraic test given to 40 science students in 1992 (Kaur & Sharon, 1994).

This paper is an extension of the 1996 analysis of the algebraic test sat by first year mathematics students. Do the same five categories still exist and are the categories consistent with a similar cohort in 1997? Are many of the weaknesses apparent in the five categories already present in senior secondary mathematics? If so, are there any automatic improvements in algebraic skills with higher learning?

## The Study

In 1997 first year students who gained between 50% and 75% in the Bursary Mathematics with Calculus examination and who enrolled in a university mathematics paper (N=540), sat a one and a half hour, 28-item algebraic test in their second week of lectures. Calculators were not permitted and the students were provided with a table of relevant formulae. Students were encouraged to display all working. A third of these students completed the test within one hour.

A week later, two senior secondary mathematics classes from two different schools (N=16, N=21) and two second year university mathematics classes (N=142, N=198) sat similar six-item algebraic tests based on the categories determined from the 1996 and 1997 first year university algebraic tests. Calculators were permitted in all these classes and the tests took about 15 minutes to complete. Again, students were encouraged to display their working.

Each question in the six-item test was adjusted to the level of the students' mathematical knowledge. For example, a second year university student would be expected to solve $m^2-8m+4=0$ as part of their work on second order differential equations, but a senior secondary student would be more familiar with the same question in terms of $x$ rather than $m$.

A comparison was drawn between the three mathematics levels: senior secondary, first year university and second year university levels. The analysis was based on a random sample of 100 scripts from each of the first and second year classes and all 37 senior secondary scripts.

### *The Categories Confirmed*

An analysis of the 1997 algebraic test for first year university mathematics students showed that in each of the five categories mentioned in the "introduction" between 19% and 29% of the first year university students gave similar types of incorrect responses. This confirmed that the five categories were still relevant in 1997.

Although there were a variety of other algebraic errors, especially in questions involving integration, differentiation and complex numbers, many of these incorrect responses could be placed in one of the five categories. Nevertheless, one other possible category was temporarily considered. Some students appeared to have difficulty understanding what the question was asking. For example, 9% of the students did not understand the meaning of "Find $\left|\frac{1}{x+iy}\right|$". Some students believed that the question was asking them to convert the denominator into a real value by multiplying by $\left(\frac{x-iy}{x-iy}\right)$. However, this difficulty was not widespread or consistent enough (>10% of students) to form a sixth category and any difficulty seemed to be limited to unfamiliar notation.

### *Understanding The Order Of Operation Agreement*

Students need to be aware of the basic order of operations in arithmetic encountered in late primary or early secondary level, and the extensions or limitations of these concepts into algebra. For example, in algebra different variables added or subtracted within a bracket cannot be combined. This means that calculation within a bracket may not necessarily be the first order of operation. The most common incorrect response in this category in the 1996 test occurred when a negative value was multiplied into a bracket containing more than one variable. Students tended to deal with brackets by ignoring them. For example, - (x + y) would become -x + y.

*Aim*: For each cohort a question was selected to specifically monitor whether every variable inside a bracket was multiplied by a negative value outside the bracket.

*Question:*

Senior secondary $\quad 3x-(\frac{1}{4}y+5z)+1=0$

First year university $\quad 1-(1+\frac{x}{2}-\frac{x^2}{3})$

Second year university $\quad 6x-2z\frac{\partial z}{\partial x}+3y-(4z-4x\frac{\partial z}{\partial x})=0$

***Results*:**

| class | correct | no answer | failed to multiply all items | other (esp rearranging) |
|---|---|---|---|---|
| Senior secondary | 0.48 | 0.12 | 0.22 | 0.18 |
| University Year 1 | 0.48 | 0.05 | 0.28 | 0.19 |
| University Year 2 | 0.51 | 0.24 | 0.17 | 0.08 |

*Table 1: Average proportion of students for each type of response to "understanding the order of operations agreement".*

Table 1 shows that approximately half the students in each of the three cohorts gave the correct answer. On average, 22% of senior secondary students, 28% of first year students and 17% of second year students made the same mistake. They ignored the brackets. These results were slightly higher than for the 1996 analysis of first year university students.

The slightly lower proportion of students making this same error in second year university (Column 4, Table 1) was expected since students weak in first year university mathematics were unlikely to pass into second year. However, almost a quarter of the second year university students did not give an answer or show any working despite having been exposed to partial derivatives at the end of the previous year.

Senior secondary students may have exhibited fewer errors than first year university students because of algebraic skills revision done as part of the secondary course a few weeks before the 1997 algebraic test was sat.

### *Understanding The Properties Of Numbers, Especially Fractions.*

Fractions are encountered initially in primary school and the processes learned at this level can impact later on algebraic skills. First year university algebra predominantly involves working with rational functions. In the 1996 algebraic test given to first year university students, the major areas of concern were in simplification and addition/subtraction of rational functions.

*Aim*: For each cohort two questions were selected. The first question was designed to monitor whether students could simplify a rational function where one element on the numerator matched the denominator. A second question monitored how successful the students were at adding two rational functions.

*Simplification Of Rational Functions:*

*Question:*

Senior secondary: $\dfrac{2x^2-2x}{2x}$

First year university: $\dfrac{ab-a^2b^2}{ab}$

***Results:***

Second year university: $\dfrac{2n+2n^2}{2n}$

| class | correct | no answer | Incorrect response A | Incorrect response B | Incorrect Response C |
|---|---|---|---|---|---|
| Senior secondary | 0.76 | 0.08 | 0.00 | 0.12 | 0.04 |
| University Year 1 | 0.76 | 0.00 | 0.01 | 0.08 | 0.11 |
| University Year 2 | 0.88 | 0.01 | 0.04 | 0.04 | 0.01 |

*Table 2: Average proportion of students for each type of response to "simplification of rational functions"*. (A, B and C refer to the type of error described below.)

Overall, 16% of senior secondary students, 20% of first year university students and 9% of second year students exhibited difficulty when simplifying a given rational functions (Table 2). The proportion of students making errors in 1997 were similar to the 1996 analysis.

*Incorrect Response* A *in Table 2*: $\frac{2n}{2n}+\frac{2n^2}{2n}=1+2n$. Students assumed that $2n^2$ was the equivalent to $(2n)^2$. There were slightly more second year university students who gave this type of incorrect response than from the other two cohorts (senior secondary and first year university).

*Incorrect Response B in Table 2*: $\frac{\cancel{2n}+2n^2}{\cancel{2n}}=n$. The element on to the left of the + has been cancelled to zero by the denominator. The proportion of students committing this error was highest amongst the senior secondary students and second highest for first year university students.

*Incorrect Response C in Table 2:* $\frac{\cancel{2n}+2n^2}{\cancel{2n}}=2n^2$. Similar to Response B except that the $2n$ on the denominator and numerator are cancelled. First year university students dominated with this type of error.

Incorrect cancellation of rational functions was a concern for senior secondary students and first year students. Since the proportion of students committing this error was lower for second year university students, this type of incorrect response may improve with higher mathematics learning.

*Addition Of Rational Functions:*

*Question:*

| | |
|---|---|
| Senior secondary: | $\frac{1}{x-3}+\frac{1}{2x}$ |
| First year university: | $\frac{1}{x+1}+\frac{1}{x}$ |
| Second year university: | $Y\ (s^2+2s-8)-1=\frac{1}{s-3}$ |

***Results*:**

| class | correct | no answer | Error in finding denominator | Error in finding numerator | other (esp rearranging) |
|---|---|---|---|---|---|
| Senior secondary | 0.56 | 0.16 | 0.27 | 0.01 | 0.00 |
| University Year 1 | 0.69 | 0.01 | 0.27 | 0.03 | 0.00 |
| University Year 2 | 0.36 | 0.17 | 0.37 | 0.04 | 0.06 |

*Table 3: Average proportion of students for each type of response to "addition of rational functions".*

All three cohorts displayed difficulty obtaining the correct denominator in a rational function (Table 3). Nearly a third of both secondary students and first year students, and over a third of second year university students, experienced this difficulty. Even if the denominator was correct, a small proportion of students in each cohort then made errors calculating the correct numerator. Again, 17% of second year students and 16% of senior secondary students chose not to attempt the question.

### *Understanding The Structure Of Obtaining A Solution.*

Basic algebraic manipulation involves keeping an equation balanced throughout a calculation and understanding the contextual meaning of the symbols being used (Teppo & Esty, 1995). In the 1996 analysis it was noted that difficulties in this category became obvious whenever students manipulated an equation, used the quadratic formula or completed the perfect square. Values could be correctly in place initially, but calculations then led to errors in manipulation.

*Aim:* To see if students, once they substitute in the correct values into a formula or equation, can continue their calculation without making errors that alter the structure of the equation.

*Question:*

| | |
|---|---|
| Senior secondary: | $x^2-8x+4=0$ |
| First year university: | $x^2-6x-15=(x+a)^2+b$ |
| | $3-8x+x^2=0$ |
| Second year university: | $m^2-8m+4=0$ |

***Results*:**

| class | correct | no answer | quadratic formula incorrect | complete perfect square, $b$ incorrect | other |
|---|---|---|---|---|---|
| Senior secondary | 0.71 | 0.03 | 0.19 | 0.05 | 0.02 |
| University Year 1 | 0.60 | 0.06 | 0.17 | 0.13 | 0.04 |
| University Year 2 | 0.54 | 0.24 | 0.14 | 0.02 | 0.06 |

*Table 4: Average proportion of students for each type of response to "structure of obtaining solutions".*

When students chose the technique to solve the quadratic equation, most (students) preferred the quadratic formula rather than completion of the perfect square technique. Only a few of the students who chose to use the latter exhibited errors in their calculation. Students who preferred the quadratic formula but were required to complete the perfect square (especially for the question given to first year university students) often made errors in calculating $b$ in $(x+a)^2+b=0$. Of those students who preferred the quadratic formula between 14% and 19% substituted the values correctly into the formula but their common errors involved either shrinking of the divisor line (8% to 12%) or miscalculation within the square root (Table 4). Shrinking of the divisor line can be illustrated in the following example: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ became $x=-b\pm\frac{\sqrt{b^2-4ac}}{2a}$.This error can also overlap the category "understanding the properties of numbers, especially fractions".

Generally, accuracy in calculations did not improve with higher learning and first year university results were similar to the 1996 analysis. Unfortunately about a quarter of the second year students did not attempt to solve the quadratic equation.

### *False Generalisation*

Students can apply a previously learned technique to a context that may appear to be similar but is actually different. The technique the students use is inappropriate for the new context (Kaur & Sharon, 1994). For example: $\frac{(1+t)^2}{t^2}=\frac{1+t}{t}$. Here, the student used subtraction of indices to this situation when the 'base' variables were different.

In the 1996 analysis with first year university students it was found that if the questions were given as "Solve (3x-4)(x+1)=0" and "Solve (3x-4)(x+1)=2", 93% solved the first equation correctly but only 57% solved the second equation correctly. Some of the students (21%) tried to use the same method as the first . That is, (3x-4)=2 or (x+1)=2.

*Aim*: To monitor whether students use inappropriate rules or techniques in apparently similar but different contexts.

*Question:*

| | |
|---|---|
| Senior secondary | Solve $(x-2)(x+3) = -4$ |
| First year university | Solve $(2x-1)(x-1) = 5$ |
| Second year university | Solve $x(x-2)(x+3) = -4x$ |

***Results*:**

| class | correc t | no answer | Incorrect x-2=-4 or x+3=-4 | Ignored right hand side | other (esp. error in expanding or factorising) |
|---|---|---|---|---|---|
| Senior secondary | 0.62 | 0.03 | 0.14 | 0.07 | 0.14 |
| University Year 1 | 0.34 | 0.06 | 0.29 | 0.03 | 0.28 |
| University Year 2 | 0.66 - 0.29* | 0.20 | 0.05 | 0.01 | 0.08 +0.29* |

*Table 5: Average proportion of students for each type of response to "false generalisation".*

* In Table 5, some of the second year university students (29%) cancelled the $x$'s on both sides of the equation and thereby ignored a possible solution of $x = 0$. This error is discussed in the next category.

The first year university students exhibited the highest proportion of incorrect responses and most of these (29%) used the same technique as they would when the right hand side of the quadratic equation was zero (Column 4, Table 5). The results were similar to the 1996 analysis when 21% of the students made the same error. Even though many of the students in second year university mathematics may have remembered this type of question from the 1996 algebraic test the previous year, 20% of these students did not attempt the question.

The dramatic increase in proportion of first year university students who made the same error (29%) could be attributed to the change in context in which such a question is given. At school, students are reminded to rearrange equations so that zero is on the right hand side. Such calculations are usually taught as an isolated topic. At university this is a skill that does not get any reinforcement and the equation is usually within a different context. In the case of the 1997 algebraic tests, the question was out of the context in which the algebraic skill would have been learned.

### *Understanding The Range of Possible Solutions*

Students need to look for solutions other than the most obvious. When students become familiar with complex numbers and functions, they need to consider complex solutions as well as real solutions. In finding the values of $x$ when $\sin x = \frac{1}{2}$ students need to consider solutions other than $0 \le x \le \frac{\pi}{2}$. When locating all critical points, students need to be aware of solutions they could be eliminating with inappropriate cancellation of variables. On the other hand, when graphing a function such as $f(x) = \sqrt{x^2 - 4}$, students often incorrectly include the negative values of $f(x)$. Therefore this category is more than just looking for all possible solutions, but more about understanding which solutions are relevant.

For specific contexts students can be 'trained' to find negative as well as positive solutions for their algebraic calculations. Most students are likely to get this type of calculation correct. For example, $(x-2)^2 = 9 \Rightarrow (x-2) = \pm 3 \Rightarrow x = -1 \text{ or } x = 5$. The question was given to a class of average senior secondary students (N=16). Ten of these students gave the correct answer while four students ignored the negative solution and gave their answer as "5".

However, in not so familiar contexts most students do not consider other possible solutions. The following question was given to the second year university students as well as to a class of accelerated senior secondary students (N=21).

*If $x > y$, is it true that $\frac{1}{x} < \frac{1}{y}$?*

***Results:***

| class | correct | no answer | negative not considered | both negative for x and y | other |
|---|---|---|---|---|---|
| Senior secondary | 0.04 | 0.01 | 0.86 | 0.08 | 0.01 |
| University Year 2 | 0.06 | 0.01 | 0.81 | 0.10 | 0.02 |

*Table 6: Average proportion of students for each type of response to "understanding the range of possible solutions".*

Most students did not consider that $x$ or $y$ could be negative as well as positive. Of the students that did consider this possibility 8% of senior secondary students and 10% of second year university students assumed that both variables could only be negative. Very few students (4% and 6% respectively) considered the possibility of $x$ being positive and $y$ negative (Table 6).

The first year university students were given a question that involved the solution of a trigonometric function. The question was similar to that given in the 1996 test and the results were comparable. Solutions outside the $0 \le x \le \frac{\pi}{2}$ range were not considered by 20% of the students.

So, although students can be 'trained' to consider other possible solutions within one context, this does not necessarily mean that the students understand what they are finding or will think of applying the same ideas to other contexts.

## Conclusion

Analysis of the 1997 algebraic test given to first year university students concurred with the analysis of a similar algebraic test given in 1996. Categories of algebraic weaknesses and proportion of students displaying those weaknesses were similar for both years. The categories "understanding the order of operations", "understanding the structure of solutions" and "understanding the property of numbers" are revised briefly in algebra at the beginning of the final year of secondary school, and are assumed by the university to be mastered before entrance to university. The other categories of "false generalisation" and "understanding the possible range of solutions" are skills acquired in senior secondary school and early university. Coping with these skills require a good grasp of the other categories as well as linkage of ideas and choice of appropriate mathematical tools.

Algebraic difficulties in most of the categories were exhibited by senior secondary school students. These difficulties became more widespread and consistent with first year university mathematics students, and slightly less widespread (in some categories) with second year university students. One possible explanation for this trend is that some students could have learned their algebraic skills at school in a isolated, surface learning fashion with little understanding or extension of concepts. Such students would have difficulty coping with situations in which they have choices in mathematics or are required to link ideas. Combine this with a four month break from mathematics before university begins and some students begin university mathematics with gaps in skills.

By the time the student reached second year university there was evidence of slight improvement in categories such as order of operations, simplification of rational functions, structure of equations, false generalisation. However, similar weaknesses still exist, probably because of emphasis on concepts and assumption of prior algebraic skills.

Students do not appear to markedly improve their algebraic skills through the learning of more advanced mathematical concepts. The evidence is that weaknesses in algebraic skills at school are unlikely to be corrected automatically at university without active intervention by the student. The basics learned in pre-algebra arithmetic and early secondary school could therefore play a crucial role in mathematical success of many students at a later date.

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