Understanding the Role of Assumptions in Mathematical Modeling: Analysis of Lessons with Emphasis on 'the awareness of assumptions'

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The purpose of this paper is to show empirically that 'the awareness of assumptions' is an effective teaching principle. For this purpose, lessons were designed and taught by the author with emphasis on 'the awareness of assumptions' for 9th Grade students in Kanagawa Prefecture. Lessons were analysed based on videotape-records, students' descriptions in the solution process and written impressions. For this group of students it was confirmed that 'the awareness of assumptions' was effective as a teaching principle.

Many Japanese students think that 'mathematics is unrelated to the real world' and 'mathematics is not useful' as the results of TIMSS (Third International Mathematics and Science Study) show. To improve such mathematical beliefs, the author thinks that the following two points are important. First, teachers should make students recognise that so called 'word problems' appeared in the mathematical textbook or taught in mathematical lessons can be related to the real world. Secondly, they should make students understand the method of mathematically solving real world situations.

The process by which we mathematically solve real world situations is usually called the mathematical modelling process (Miwa, 1986a). Typically the process consists of the following four steps: (1) Formulation, (2) Mathematical works, (3) Interpretation and evaluation, (4) Improvement of model. Step (4) usually involves repeating steps (1) and (3) several times. The most important stage in this process is the formulation stage. At the formulation stage, "assumptions based on deliberate suppression or neglect of irrelevant details are set up" (Miwa, 1986a, p. 401). If these assumptions are not appropriately set up, the nature of the situation is distorted, and the problem cannot be solved appropriately. The setting up of appropriate assumptions can be considered as the most important thing in performing mathematical modelling. Therefore, it is necessary to make students understand the role of assumptions in mathematical modelling.

Up to now, the author has proposed 'the awareness of assumptions' as a teaching principle to make students understand the role of assumptions (Seino, 2004a, 2004b). 'The awareness of assumptions' as a teaching principle means to attach great importance to the process of setting up assumptions, to make students become aware of (tacitly) setting up assumptions, and to examine particular assumptions closely. The purpose of this paper is to show empirically that promoting 'the awareness of assumptions' is an effective teaching principle. For this purpose, lessons were designed and taught by the author with emphasis on 'the awareness of assumptions'. Moreover, effectiveness of 'the awareness of assumptions' was analyzed based on the lesson's videotape-records, students' descriptions in their solution process, and students' impressions written in about ten minutes immediately after the lesson.

The Construction of the Lesson Based on 'the awareness of assumptions'

Roles of 'The Awareness of Assumptions' and Stages in which Assumptions are Recognised or Implied

There are two aspects in 'the awareness of assumptions'. The first aspect is a teaching principle as mentioned above. The necessity and the importance of making students become aware of assumptions having been pointed out so far (e.g., Pinker, 1981; Miwa, 1986a; Shimada, 1990). For example, Shimada (1990) points out the following:

> In typical current application examples, we considered tacitly the hypothesis that existed in the base as admission, and not became aware of it as assumption. Therefore, students could not help holding recognition that the mathematical problem had been away the reality…It will be useful to activate this kind of class if students become aware of the condition concerned with reality that is behind answer of problem, and consider its suitability (p.45, translated from the Japanese).

In a word, 'the awareness of assumptions' plays the role of a bridge that connects the 'real world' to the 'mathematical world'. This is the first role.

Moreover, another aspect is a method of performing the problem solving. This aspect is seen from the students' side. As to this aspect, Miwa (1986b) points out:

> To be aware of setting up assumptions and improving assumptions are the key points to letting students participate in the developing process of knowledge. (p.25, translated from Japanese)

That is, 'the awareness of assumptions' promotes activities that reflect on the formulation stage in mathematical modelling process (Figure 1). Therefore, it is possible to develop the foundation of mathematical modelling through this way of teaching. This is the second role.

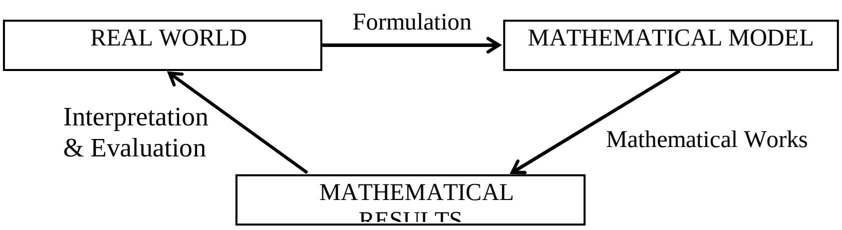

Figure 1. The process of mathematical modelling (Miwa, 1986a, p.402).

The next question is at what stages in the mathematical modelling process are assumptions recognised or implied? It is clear that assumptions are set in the 'Formulation' stage. However, the stage in which assumptions are recognised or implied is not necessarily one stage. The author (Seino, 2004b) identified these stages based on the thought process of the modeller. And, when the author identified that, the author referred to the roles of assumptions which Galbraith and Stillman (2001) addressed. Finally the author set framework based on the following three aspects:

A. Assumptions recognised or implied in the formulation stage
B. Assumptions recognised or implied in the stage of mathematical works
C. Assumptions recognised or implied in the stage of interpretation and evaluation of mathematical conclusions toward real world

Aspect A is used or implied when the modeller formulates situations. The role of Aspect A is to connect 'real situations' to the 'mathematical model'. Aspect B is used or implied when the modeller cannot perform mathematical work, or when the modeller intends to simplify mathematical work. The role of Aspect B is to make mathematical work possible, and to simplify mathematical work. Moreover Aspect C is used or implied when the modeller interprets and evaluates mathematical results. The role of Aspect C is to give the opportunity of better modelling. This lesson especially focuses on Aspects A and C in the framework.

The Construction of the Lesson Based on 'The Awareness of Assumptions'

Teaching material used in this lesson is 'the catching-up problem' a typical textbook problem, and 'the catching-up problem based on real data' (Figure 2). These two problems have the same structure. A different point is whether the speed is given as a condition or not. The real data were based on two videos taken simultaneously by the author as a walking person and author's friend as a cycling person. The person walking held a GPS (Global Positioning System) to measure the distance. The distance data and time data were shown in the video for students at the lesson (including a stop at a traffic signal).

The catching-up problem in the textbook: *The younger brother left the house for the station 2km away. The elder brother ran down the same road on his bicycle ten minutes later. If the speed of younger brother's walking is 80m per minute, and the speed of elder brother's bicycle is 240m per minute, how long does it take the elder brother to catch up with the younger brother assuming he left 10 minutes later?*

The catching-up problem based on real data: *The younger brother left the house for the station 2km away. The elder brother ran down the same road on his bicycle ten minutes later. How long does it take the elder brother to catch up with the younger brother assuming he left 10 minutes later? Predict this based on the following data. The data was acquired by measuring the distance between actual utility poles.*

1. Data concerning time and distance after the younger brother leaves

Distance(m)	0	55	98	144	190	231	277	341	398
Time(second)	0	41	68	98	126	152	181	221	257

2. Data concerning time and distance after the elder brother leaves

Distance(m)	0	55	98	144	190	231	277	341	398
Time(second)	0	20	30	40	50	60	70	83	97

Figure 2. The problem and data.

In this lesson, first, 'the catching-up problem in the textbook' is presented by the teacher (the author). After students answered the problem, the teacher explicitly asked students to consider tacitly assumed assumptions. Examples of this are: 'the speed is constant' and 'he doesn't stop on the way'. These assumptions correspond to C in the framework because students reflect on the problem in interpretation and evaluation stage. There were two purposes in this approach. The first was to make students recognise the condition in this problem is an assumption. The second purpose was to make students realise the question of 'actually, does an assumption that the speed is constant function effectively?'

Secondly, 'the catching-up problem based on real data' was presented. To solve this problem, the assumption of 'thinking the speed to be constant' is needed. This assumption corresponds to A in the framework. After students thought about the problem, the teacher asked students to present the results of their ideas to the class. Then, the teacher asked students to make explicit their assumptions. The purpose in here was to make students evaluate the importance of the assumption of constant speed.

Thirdly, the teacher questioned the necessity of better modeling, and made the students aware of other assumptions they have to consider (e.g., traffic lights). This is corresponds to C in the framework.

Finally, the real situation which assumes the speed to be constant (or near-constant) was presented to the students. The purpose here was to make students appreciate the necessity and importance of setting up assumptions, and to make students appreciate the usefulness of mathematics.

The Lesson Based on 'the awareness of assumptions'

Description of the Lesson

Lessons were taught for 9th Grade students in a private junior high school in Kanagawa Prefecture (three classes). The number of students in the three classes were each 19 students (class-A), 19 students (class-B), and 18 students (class-C). These lessons built on prior knowledge acquired in 8th Grade (e.g., understanding of linear functions). All lessons were done in 45minutes. The lesson discussed here is class-A's lesson. In the following sections, the lesson is described based on videotape-records.

First, 'the catching-up problem in the textbook' was presented to the students. Many students were able to tackle this problem at once. Next, the teacher asked the following question to the students. 'When you assume this problem is a real situation, do you think that the obtained solution is appropriate?' Moreover, the reason was questioned. As soon as the teacher questioned students, they questioned the teacher conversely: 'Do we assume that there is nothing on the road?' and 'Can you say that the solution was appropriate, if it was up to how much time?'

Student 1: Do we assume that there is nothing on the road?
Teacher: Well?.
Student 1: Or, do we think realistically? It differs according to whether we think realistically.
(Another student said that it differed according to whether we think about the initial speed.)
Teacher: We think realistically. (It began to be noisy as soon as this word was said.)
Student 1: Then, it is impossible.
Student 2: Can you say that the solution was appropriate, if it was up to how much time?
Teacher: Do you judge it is appropriate, if it is up to how much time?
Student 3: just to meet.
Student 4: 2 or 3 minutes.

Nine students thought that the obtained solution was appropriate. And, nine students thought that the obtained solution was not appropriate. One student said that he could not answer the question. Moreover, many assumptions concerning this problem were clarified in various discussions (see Table 1).

Table 1
Assumptions Clarified by Students

Assumptions		
no traffic signals	There are no friends on the road	The road is straight
no resistance	They don't decelerate in the curve	Traffic is light
The speed is constant	They don't become tired	no acceleration

After students clarified various assumptions, the task ('the catching-up problem based on real data') and a route map were presented to the students. In the map, the route taken on foot and on bicycle was drawn. Then, an activity was set by the teacher to calculate the speed by using data taken from the video, and to predict how long it took the elder brother to catch up with the younger brother assuming he left 10 minutes later. Afterwards, the video showing a person walking and a person cycling was projected onto the screen. Then, the teacher presented the graph of both data concerning speed and distance.

In this lesson, two students presented their ideas. The author shows only one student's ideas.

> Student T.J: First, the average speed of the younger brother and the elder brother in each position was calculated. Then, the younger brother's average speed became about 1.4875m/s. That is, it was about 89m per minute. With the elder brother also calculated similarly, it was about 221m per minute. So, when the elder brother starts, the younger brother's position becomes 89m/minute × 10 minutes =890m. Next, well… because I want to derive time that the elder brother caught up, I divide 890m by the difference of the speed of the elder brother and the younger brother. And, when the time that the elder brother left is added and calculated, it becomes roughly 16.7 minutes. Moreover, when I multiplied 16.7 minutes by 89m/minute, which was the younger brother's speed, I think that elder brother catches up with younger brother at 1486.3m (student receives a round of applause).

After this, the discussion concerning the formulated speed in other words 'the speed is constant' was done. Here, the teacher clarified what assumptions were set up; how the elder brother caught up the younger brother was projected onto the screen, and students compared actual position and the result of their calculation. The elder brother caught up the younger brother at 1430m, 15minutes 30seconds later. Afterwards, Figure 2 was presented and the usefulness of assumption of 'thinking the speed to be constant' was confirmed. Moreover, the teacher tried to make students appreciate the necessity of better modeling.

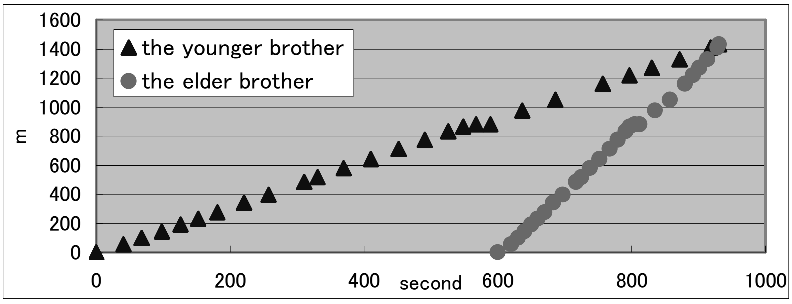

Figure 2. Graph of data concerning speed and distance obtained from video: by foot and by bicycle.

Teacher: This figure shows where all times and distances were plotted. The vertical axis is meters in distance, and the horizontal axis is seconds in time. The speed?

Student 5: It is almost constant.

Teacher: It is almost constant. What is happening here? (the teacher indicates the points at about 900m for the younger brother and the elder brother) Why is the parallel?.

Student 6: It is place in which the elder brother and the younger brother stopped.

Teacher: It represents stopping at the traffic signal. When we take into consideration the traffic signal, we can obtain more accurate values.

Student 7: Oh.

The teacher confirmed that students made the best use of the assumption of 'thinking the speed to be constant' in daily life.

Analysis of Students' Worksheets

Then what strategies were used by students in formulating the speed in these lessons? When the author analysed all of the students' worksheets in the three classes, the speed was formulated by the following five strategies:

- It pays attention to the last data, and it is formulated at distance ÷ time.
- It pays attention to calculating the speed each data point using, and an average speed was formulated.
- It pays attention to taking the difference of each consecutive distance (e.g., distance2-distance1) and dividing this by the difference of each equivalent pair of times (e.g., time2-time1). E.g., $(d_2-d_1)/(t_2-t_1)$, $(d_3-d_2)/(t_3-t_2)$, An average of these calculations was also formulated by the students.
- Data plotted on the graph is formulated by assuming a linear relationship.
- Others

The numbers of responses that correspond to each category were respectively: 27, 18, 23, 4, and 2. The total number of responses is 74 since 21 students offered two formulations. Overall there were rremarkably few responses given. Importantly, when students formulated the speed, most students chose not use a graphical representation.

Discussion

Understanding the Role of Assumptions in Mathematical Modelling

In this paper, the author does not insist on always making students become aware of assumptions in solving word problems. Simply making students become aware of assumptions is not enough to understand the role of the assumptions. Teachers need to make students appreciate the effectiveness and reasonableness of clarified assumptions. In this lesson, the author presented 'the catching-up problem in the textbook' and 'the catching-up problem based on real data' as a pair. The effectiveness of assumptions was actualised more by using two related problems. The author confirmed the validity of this teaching principle from the following students' written impressions.

> At first, I doubted the answer of the problem obtained from the assumption that 'the speed was constant'. But, when I saw an actual value, it was almost constant. So, I understood that assumption of 'the speed was constant' was appropriate (Student N.Y).

> I have never thought it about 'the speed is constant or not' and, 'the speed changes depending on the traffic signal'. I think that I can review a variety of mathematics problems by such an idea. I want to review problems I answer next time. (Student Y.K)

In addition to these students, many students have described the usefulness of ‘thinking the speed to be constant’ and the necessity and importance of assumptions. In this way, the author can think that making students become aware of assumptions and making students reflect on assumptions is important for understanding the roles of assumptions.

Moreover the author can demonstrate the transformation of student’s view for mathematical textbook problems and mathematics through this lesson with emphasis on ‘the awareness of assumptions’. Actually, students described their impressions as follows.

> We usually solve such problems by accepting the speed as constant, but I thought that the speed changed due to various conditions actually. However if the subjects don’t stop, we can consider the speed as almost constant. So I roughly understand that mathematical problems were real. Up to now, I have been solving mathematical problems while doubting those realities. But if conditions are suitable, I understood those problem were real. Therefore I think I can solve mathematical problems without doubt from now on. (Student Y.K)

> When I solved similar mathematical problems before, I thought, ‘Did the solution actually fit real situations?’. But, when I verified the solution for real situations, I understood it almost showed actual values. (Student T.T)

> I now understand the necessity of using mathematics for real situations. (Student H.A)

To transform students’ view for word problems, it appears to be important to offer the opportunities that review word problems students have solved once. It can be thought that the presentation of ‘the catching-up problem in the textbook’ was effective in this sense. Moreover, if students usually solve word problems that assumptions and the solution are not suitable for real situations, the students’ view for word problems should not change. The teacher should improve them if there are unrealistic word problems.

Transformation of Students’ Recognition for the Usefulness of Graphical Representation

When students formulated the speed, they almost all did not use graphical representation. However, 4 students out of 56 (total of three classes) formulated the speed by using graphical representation. When the description of one of these students’ impression is seen, it is understood that having drawn the graph greatly contributes to problem solving. That is, the student recognised the usefulness of graphical representation, and assumption of ‘thinking the speed to be constant’. Moreover, it appears also to other three students’ impressions that drew the graph. Here, the author shows only one student’s impression.

> I have thought the speed was not constant. But when I saw the graph, I understood that I could consider the speed to be almost constant, because it had only a few errors. (Student N.H).

On the other hand, ten students said they could appreciate the usefulness of a graph through this lesson. For example, students described their impressions as the following.

> We can see the speed of both the bicycle and on foot with graphical representation. When I connect plotted points, the line is almost straight. And, when younger brother and elder brother stopped at a traffic signal, the drawn line was parallel to the x axis. So, I recognised the accuracy of a graphical representation. (Student N.Y)

> The speed is never constant. But, when I drew a graph, the line was almost straight. I have understood that the graphical representation is more useful than the expression according to the cases. (Student N.H)

In constructing a mathematical model for this case, a graphical representation illustrates the assumption of near-constant speed, and also shows students that this assumption is based in reality.

Concluding Remarks

Recognising the usefulness of mathematics is one of the most important aims in school mathematics. Teachers try to make students recognise it in some way by using realistic problems. But using realistic problems is not enough. Therefore, we have to turn our eyes toward key components in mathematical modelling to accomplish genuine mathematical modeling. That is the role of assumptions.

In this paper, the author suggested one way of making students understand the role of assumptions, namely through 'the awareness of assumptions'. The effectiveness of this approach is confirmed through analysing of lesson videotape-records, students' descriptions in their solution process and students' written impressions. That is, students who were able to appreciate the usefulness of mathematics and to understand the role of assumptions were identified, and for this group of students 'the awareness of assumptions' was effective as a teaching principle. Moreover, the following two findings were obtained. First, assumptions can be recognised or implied by students not only in the formulation stage but also in the interpretation and evaluation stage. Secondly, when students solve real world problems, most students chose not to use graphical representation. However, when the author studied students' written impressions, it was clear that students recognised the effectiveness of graphical representation. Students were able to say, 'But when I saw the graph, I understood that speed is almost constant'. In this way, the assumption of (almost) constant speed became a realistic or reasonable assumption for students.

For students who are not accustomed to mathematical modelling, the teacher's main role is to clarify assumptions, and to inform students of the importance of assumptions. Understanding the role of assumptions and what constitutes a reasonable or realistic assumption are important outcomes from this study.

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