

Problem Solving in Algebra: The Role of Arithmetic Thinking and Affective Factors

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In this paper we discuss the roles of, and the relationship between, arithmetic and algebraic thinking in algebraic problem solving. We report the results of a study where we recorded students' mode of thinking, problem solving ability, mathematical anxiety and self-concept. We found a positive correlation for girls between anxiety and problem solving performance, but for boys the positive correlation was with self-concept. We also found a reluctance on the part of students to use algebraic thinking.

Background

Many students have had difficulty with algebraic problem solving. There have been a number of studies conducted by mathematics education researchers to attempt to understand the nature of these difficulties. Kieran (1992) sees generating equations from words as the major area of difficulty for high school algebra students. Much research (e.g. Lochhead, 1980; Clement, Narode and Rosnick, 1981; Kaput & Sims Knight, 1983; MacGregor & Stacey, 1993; Thomas, 1994) has analysed the difficulties faced by students while solving algebra word problems and have offered suggestions to overcome these difficulties. Brown *et al* (1988) expressed the opinion that students appear to cover their inability to translate the relationship between the words into symbolic form or to solve resulting equations resorting to memorising rules and procedures that they eventually come to believe represent the essence of algebra.

Others have tried to compare arithmetic and algebraic modes of thinking in the context of problem solving. Filloy (1985) stated that algebra is the continuation of arithmetic. Kieran (1992) delineated arithmetic and algebraic thinking by referring to arithmetic operations carried out on numbers to yield numbers as procedural and a set of operations carried out on algebraic expressions as structural. While differentiating two types of operations, Kinch and Greeno (1985) see the difference as mainly due to the complexity of algebraic schemata compared with arithmetic set schemata. In the same context Herscovics and Linchevski (1994) stated, from a study where students solved fifty algebraic equations of different types, that a cognitive gap exists between arithmetic and algebra which is characterised by the students inability to operate spontaneously on the unknown. There is evidence emerging (Thomas, 1994; Crowley, Thomas & Tall, 1994) that many students, described as *process-oriented*, are locked into an arithmetic, procedural way of thinking even when doing algebra.

Affective Factors

Problem solving can be seen with a different perspective, as an interaction between cognitive and affective variables. McLeod (1989) claims that the cognitive processes involved in problem solving are particularly susceptible to the affective domain. From a Piagetian perspective this is not surprising since the process of acquisition of knowledge involves the constant interaction between the learning subject and his or her environment, and affective variables result from this same interaction. Boekaeris *et al* (1995) stated that mathematical problem solving can be seen as a process in which cognitive and affective variables interact and Lester (1994) has described this interaction as involving several closely related independent categories of factors such as knowledge acquisition and utilisation, beliefs, affects and socio-cultural contexts. Hart (1989), too, summarised the

affective domain as comprising a wide variety of beliefs, attitudes and emotions including anxiety which typically contain a strong emotional feeling. Self-concept, a reflection of how one sees oneself, is also part of the affective domain. Skaalvik and Rankin (1995) define self-concept colloquially as a composite view of one self, the hierarchical structure of which Shavelson, Hubner and Stanton (1976) and Shavelson and Bolus (1982) have modelled emphasising its multidimensional nature. Self-concept and anxiety have been taken in some mathematics studies as more specific components of attitudes and shown to have direct, consistent positive and negative correlations with mathematics achievement which are also gender specific. Reyes (1984), for example finds consistent positive correctional association between self-concept and academic achievement and Marsh (1993) demonstrated the gender differences in self-concept. Similarly the effect of mathematics anxiety on mathematics achievement has interested researchers, with Richardson and Woolfolk (1980) mentioning that certain features, such as an emphasis on problem solving, logic, etc. particularly provoke anxiety in students.

However the relationship between self-concept and anxiety in the affective domain and student word problem solving ability in algebra, an area of particular difficulty for students, has not been a specific area of study. This paper documents the start which we have made in looking at the importance of affective factors such as these on students' mathematical problem solving ability.

Method

Subjects

The data for the present study was collected from 116 form 3 and 4 students (age 13 and 14 years) comprising 48 boys and 68 girls. The students belong to average ability groups from one boys, one girls and one co-educational school in the Auckland region of New Zealand, and all were reported by their teachers to be following the New Zealand mathematics national curriculum.

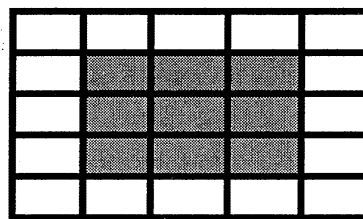
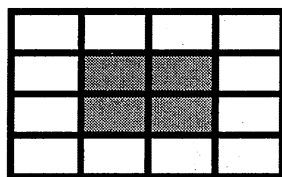
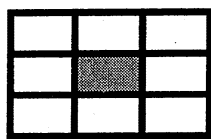
Instruments

The mathematics anxiety scale of Fennema & Sherman (1976) served as the students' mathematics anxiety measure. Fennema & Sherman scales have been among the most popularly used scales for about two decades and recently Melancon *et al* (1994) reported favourably regarding the validity of the scores obtained using the scales. The mathematical self-concept scale developed by Gourgey (1982) and having 27 items was used to measure mathematical self-concept. The reason for using this scale was that the definition given by Gourgey for self-concept constitutes a general view of the affective domain and the individual subjects report regarding their perceived capability in performing mathematics. Both scales are in a 5-point Likert format.

In addition to the attitude scales, three parallel algebraic tests of 5 problems each were administered to measure the students' problem solving ability. Examples of the problems from the three tests used are:

1. Julia had \$560. After buying a gold chain she had \$151 left. How much did she pay for the gold chain ?
2. The number of girls in the class is 12 less than the number of boys. The total number of students in the class is 34. How many girls are there in the class ?
3. To hire the pizza house for a birthday party costs a basic rate of \$70 plus \$3.50 per person. If the total bill is \$252.00, how many people attended the party ?
4. If the total amount of money collected by selling 230 tickets to the school play was \$764 and the cost of the ticket was \$4 per adult and \$3 per child, how many adult tickets and how many child tickets were sold?
5. A square garden is surrounded by square paving stones. Where N = the number of paving stones and L = the length of the side of the garden.

Complete the table to find the number of paving stones for a garden 4 long and 100 long. Write an algebraic equation to find the number of paving stones for a garden L long.



Length of the side of the garden	L	1	2	3	4	...	100	...	L
Number of paving stones	N	8	12	16					

These problems were selected to correspond to examples in the published New Zealand mathematics curriculum for each of the forms 1 to 5. It was arranged that they have the same achievement objectives and difficulty level as described in the mathematics curriculum followed in the schools. Problems were taken from levels both above and below that of the students so that there would be questions that were genuine problems for the students, being outside their experience, and we would be able to monitor the methods they used to tackle these novel problems.

The instruments were administered under the supervision of the mathematics teacher concerned within a 1-week period during the middle of the 1995 academic year. To ensure the uniformity of administration the teachers were provided with clear instructions and directions.

Results

Affective Factors

The results of the 116 students, 48 boys and 68 girls, who completed all three algebra tests and both affective factor scales were analysed for the influence of these factors. The individual students' scores on each of the three algebra tests we used, as described in table 1, show a fairly high level of correlation between the tests. Since they were designed to be of a similar content and standard this is what we would expect. We conclude that there is a fairly stable level of attainment for each student over the tests, and thus together they constitute a fairly reliable measure of ability on these types of algebra word problem. The results of the ANOVA show that there is, however, a significant difference in the overall results of the three tests ($df = 2$, $F = 31.65$, $p < 0.0001$). Analysis of this difference showed that overall the results on test 3 were significantly lower than those on the other two tests. Also, on two of the tests the girls performed significantly better than the boys (Test 1: $t = 0.943$, n.s.; Test 2: $t = 1.892$, $p < 0.05$; Test 3: $t = 2.177$, $p < 0.05$).

Table 1: Correlations for individual male and female student scores on the three tests

	Overall	Female	Male
Test 1 v Test 2	0.603	0.677	0.488
Test 2 v Test 3	0.482	0.567	0.364
Test 1 v Test 3	0.696	0.756	0.658

An analysis of the correlations between the students' individual algebra test scores and their affective factor scores on the anxiety and self-concept scales used revealed a significant positive correlation between self-concept and the results of all the tests (Test 1: $r = 0.422$, $p < 0.01$; Test 2: $r = 0.306$, $p < 0.01$; Test 3: $r = 0.394$, $p < 0.01$; $df = 114$). This seems to indicate that those who have a higher level of self-concept generally perform at a

higher level on the algebra tests, and vice-versa, confirming the finding of Reyes (1984) study, but in the domain of algebra problem solving.

Gender Considerations: When we divide the students' results into male and female we see some very interesting results emerging which tend to be masked when the results are taken as a whole. As table 1 shows, the correlations between the algebra tests based on individual scores, on a gender basis, are higher in each case for the female students than for the male students. Thus the females perform more consistently than the males across the algebra tests. When we look at the correlations between the algebra scores and the anxiety and self-concept measures on a gender basis we obtain the figures in table 2.

Table 2: Correlations between algebra, anxiety and self-concept scores for individual male and female students

	Female (N=68)	Male (N=48)
Test 1 v Anxiety	0.453	0.034
Test 2 v Anxiety	0.393	0.258
Test 3 v Anxiety	0.340	0.140
Test 1 v Self-Concept	0.415	0.472
Test 2 v Self-Concept	0.366	0.291
Test 3 v Self-Concept	0.327	0.530

For the girls every one of the correlations is significant ($p < 0.01$, $df = 66$), but for the boys, none of the anxiety results are significant, but all the self-concept ones are (Test 1: $p < 0.01$; Test 2: $p < 0.05$; Test 3: $p < 0.01$; $df = 46$). We also see clearly that there is a higher positive correlation each time between algebra performance and anxiety for the female students, and on two occasions, a higher correlation between algebra performance and self-concept for the males. When we looked at the overall differences between the scores for anxiety and self-concept between the genders we found no significant difference in scores (anxiety, $df = 99$, $t = 0.653$, $p < 0.515$ and self-concept: $df = 94$, $t = 1.152$, $p < 0.252$). Thus, while there is no significant difference for these two factors between the genders, it appears that girls who are anxious about mathematics are more likely to perform better in the algebra problem solving whereas this is not true for boys. Likewise both girls and boys who have a higher self-concept are more likely to perform better. The results on anxiety are, at first, surprising, since they are different from the negative correlations published in the research literature (see e.g. Aiken, 1976; Betz, 1978; Adams & Holcomb, 1986; Gliner, 1987; each of whom has established consistent, negative relationships between mathematics anxiety and performance) and one might think that those who are anxious about mathematics would not perform well (and for some boys this may be the case). However, the results show that the opposite is true for the girls, with their anxiety apparently motivating them to perform better. It appears that there is also an age factor involved since when we looked at the girls and boys at single sex schools in our sample, we obtained the correlations given in table 3:

Table 3: Correlations between algebra, anxiety and self-concept scores for individual male and female students aged 13 and 14

	Girls Age 13 (N=25)	Girls Age 14 (N=28)	Boys Age 13 (N=26)	Boys Age 14 (N=19)
Test 1 v Anxiety	0.873	0.481	---	---
Test 2 v Anxiety	0.675	0.243	---	---
Test 3 v Anxiety	0.403	0.013	---	---
Test 1 v Self-Concept	0.702	0.129	0.369	0.759
Test 2 v Self-Concept	0.674	-0.173	0.310	0.447
Test 3 v Self-Concept	0.574	0.070	0.327	0.455

Thus for the younger girls both anxiety and self-concept correlate significantly with algebra problem solving performance in every test, but within just one year this effect has diminished to the point where it has disappeared. Conversely, as the boys age a year from 13 to 14, their self-concept, which at 13 is not significantly correlated with performance, becomes significantly correlated a year later.

Arithmetic and Algebraic Thinking

It can be seen from tables 4 and 5 that, for the first three problems, where although algebraic methods may be useful they are not essential for their solution, an arithmetic approach is easily the preferred option for students of both ages.

Table 4: Percentages of male and female form 3 (age 13 years) students using arithmetic, trial and error and algebraic methods.

Problem	Girls (N=43)			Boys (N=40)		
	Arithmetic	Trial & Error	Algebraic	Arithmetic	Trial & Error	Algebraic
1	61.2	0.8	0	48.3	2.5	1.7
2	32.6	12.4	10.9	25.0	10.8	5.0
3	43.4	4.7	2.3	36.7	4.2	5.0
4	3.1	41.9	5.4	5.8	20.0	7.5
5	9.3	1.1	26.7	0	0	26.2

When arithmetic thinking is not successful then the students first look to trial and error, as may be seen from questions 2, and particularly, question 4, which cannot easily be solved arithmetically and in which we deliberately avoiding giving any direction towards an algebraic method of solution. In question 5, which also cannot easily be solved arithmetically, there was a slight hint of algebra given, with the introduction of a letter for a variable. This question also had a diagram.

Table 5: Percentages of male and female form 4 (age 14 years) students using arithmetic, trial and error and algebraic methods.

Problem	Girls (N=41)			Boys (N=32)		
	Arithmetic	Trial & Error	Algebraic	Arithmetic	Trial & Error	Algebraic
1	71.5	0	2.4	61.5	0	3.1
2	35.0	2.4	14.6	31.3	1.0	3.1
3	53.7	1.6	4.9	53.1	0	2.1
4	5.7	24.4	5.7	5.2	20.8	1.0
5	4.9	1.2	54.9	0	0	14.1

Interestingly, by the age of 14 significantly more girls ($\chi^2=24.0$, $p<0.001$) than boys are using algebraic methods to solve question 5. This may indicate that the girls are progressing from arithmetic to algebraic thinking earlier than the boys. The two groups comprise different students though, so this result needs further investigation, tracking the 13 year old group through to form 4, and this we will look at when they are a year older.

Conclusions

What do these results show? Summarising the results on affective factors we may say that anxiety and self-concept have a positive effect on the problem solving performance of girls, especially younger ones, whereas for boys the positive effect on their problem solving ability is self-concept, and then only for the older students. Further research is needed to see if these effects are only present in single sex schools, as described here, or whether they exist in co-educational schools as well. When doing algebra, we see that even after two or three years, if the relation between the variables in

a problem is given in the verbal form, the students show a tendency to use an arithmetic mode of thinking. A failure to solve the given problem using these arithmetic methods leads them to follow intuitive methods such as guess and check or trial and error rather than algebraic ones even though these methods are known to them and they were able to use algebra on the final question. A greater number of the students could express the relationship between the unknowns in an algebraic form if the problem was given with a geometrical figure, even when the problem was totally unfamiliar to them. Clearly the transition from arithmetic to algebraic thinking takes time, and just as there is a gap between acquisition of knowledge, understanding and application it can be seen that there is a gap between learning algebraic methods and applying them to solve problems. This transition may take less time for girls than for boys. There appears to be a relationship between the changing effects of anxiety and self-concept and the changing preference for algebraic methods over arithmetic ones. For the girls, as the correlation between these affective factors and performance disappears their use of algebraic methods increased. However, for the boys, as the correlation between self-concept and performance increased, their use of arithmetic methods increased. The underlying reasons for what appears to be a relationship are not clear and will be the subject of continued investigation.

It may be that the reluctance to use algebra can be lessened by using geometric figures to express the relationship between unknowns simultaneously with verbal forms. That mathematical thinking may be encouraged by the use of visualisation encouraged by multiple representations is well known. Thomas (1995) has suggested a theory of *cognitive integration* which may begin to explain the mechanisms by which this may take place.

Further, our results may be consistent with an information processing model of cognition, such as that explained by Silver (1987), in which there are two kinds of information processing, automatic and controlled. Automatic processing involves activation of cognitive nodes, but under the control of a particular input for working memory (internal or sensory) rather than the subject's control. In controlled information processing a sequence of nodes is activated by a controlling subject. In mathematics, by the time student reaches secondary stage, solving problems using arithmetic procedures has become automatic, though it started as a controlled process at primary school level. For this reason children might have the tendency to use arithmetic procedures for some time, even after they formally learn algebraic methods. There is a possible role played by both affective factors, anxiety for girls and self-concept for boys in this transition from an arithmetic mode of thinking to an algebraic one, or from automatic to controlled processing. Further empirical studies are needed to clarify whether the apparent time lag between the learning of algebraic methods and the application of them to solve problems does fit such a theoretical model, and, if it does, to assist in analysing possible reasons (either cognitive or affective) for this time lag.

It remains to find out if the algebraic methods ever become the preferred method of solution for students and if so at what age and for which students. Further, do affective factors such as anxiety, self-concept and other social factors influence the transition in their thinking and if so, how and why?

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