Children's Representations and Conceptual Understanding of Number

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This paper reports the developmental phase of a 2-year longitudinal study of 120 Grade 2 children, which is designed to explore the relationship between their representations and understandings, of key number processes. The study focuses on the structure of children's representations of estimating, counting, grouping, re-grouping, partitioning and multiplicative processes. The research methodology, including videotaped interview tasks used during trialing in March and April 1996, and preliminary findings will be presented for discussion and critical analysis.

Recent studies have focused on the importance of children's construction of number concepts and processes examining specific areas such as counting, addition and subtraction, or multiplication and division as separate areas of study. This research has provided insight into the diversity of children's numerical strategies but it has not examined closely, the relationship between children's representations of number and their development of number concepts and strategies. Few studies have provided an in-depth analysis of children's representations of essential and related features of number such as estimating, counting, grouping, regrouping and partitioning. Furthermore, there is growing need to examine children's construction of number concepts and strategies over time through longitudinal studies that monitor growth and change.

Studies focusing on numeration have primarily examined children's counting and grouping strategies (Bednarz & Janvier, 1988; Denvir & Brown, 1986a,b; Fuson, 1990; Hiebert & Wearne, 1992; Kamii, 1989; Steffe, Cobb & Richards, 1988; Steffe, 1991; Wright, 1991). Although counting has been the focus of much of the work on early number learning, other researchers have directed attention to the development of multiplicative reasoning (Confrey, 1994; Mulligan & Mitchelmore, in press). Rubin and Russell (1992) asserted that children's counting, grouping, estimating and notating skills are essential elements in developing representations of the number system. They described these elements in terms of "landmarks in the number system". These landmarks appear to be related to additive structure, multiplicative structure, the generation and analysis of mathematical patterns and mathematical definitions. Rubin and Russell also suggested that people who are adept with number operations e.g. computing, comparing, and estimating, have a non-uniform view of the whole number system.

Other recent studies analysing the growth of number concepts and processes, indicate that children's representations of problem-solving situations are closely linked to their conceptual understanding and the way they construct mathematical relationships (Davis, 1992; Davis, Maher, & Noddings, 1990; Goldin & Herscovics, 1991; Hiebert & Wearne, 1992; Maher, Davis & Alston 1993; Mulligan, 1996). Researchers have also highlighted children's representations as evidence of the idiosyncratic and creative ways in which they structure mathematical relationships (Thomas & Mulligan, 1995; Thomas, Mulligan & Goldin, 1994).

Some researchers suggest that conceptual understanding is built on the notion of constructing connections between representations of mathematical ideas. This involves building relationships between quantities that are represented physically, pictorially, verbally and symbolically (Hiebert & Wearne, 1992). Moreover, they suggest that building connections between external representations supports the growth of internal conceptual understanding and more coherent internal representations of mathematical ideas.

However it is not entirely clear how children use their representations in building conceptual understanding. If children's representations of their counting and computational strategies are a reflection of their developing concepts, or if their representations actually enhance the development of mathematical concepts and processes, then their representations may provide valuable evidence of how conceptual understanding grows.

The aim of the present study is to analyse and describe, in the greatest possible detail, children's internal representational capabilities evidenced in solving tasks focused on the structure of the number system and related numerical processes. At the same time, the study examines the relationships children form between various numerical processes such as counting, grouping, additive and multiplicative strategies. A 2 - year longitudinal study will provide a more coherent picture of how young children construct numerical representations, and whether these representations change and develop to accommodate more complex numerical structures over time.

Imagery in Representations of Number

The role of imagery in the representation of mathematical ideas has been highlighted in recent studies investigating representations of number. In the past, personal visuo-spatial representations of number (number-forms) were first described by Galton (1880). Seron, Pesenti, Noel, Deloche and Cornet (1992) studied adults' representations of number claiming that "some automatically 'see' the numbers they are confronted with in a precise location in a structured mental space, others 'associate' specific colours with given numbers" (p.164). A wide range of representations were found categorised into two main types: continuous lines, scales, grids (coded as number forms), and coloured codes. Associated images and simple analogical representations were also observed. Most subjects asserted that their images of number emerged between 5 and 8 years of age, or that their representations were a direct result of teaching. They concluded that number-forms are used to code the number sequence, and that the function of this phenomenon should be examined in numerical contexts and in number operations and calculations.

In a cross-sectional study of 166 children (K-6) and 79 high ability children (Grades 3-6) it was found that the children's internal representations of numbers were highly imagistic, and that their imagistic configurations embodied structural development of the number system to widely varying extents, and often in unconventional ways (Thomas, Mulligan & Goldin, 1994). Children's active processing of internal images were found to be static or dynamic in nature: static meaning a fixed representation, and dynamic as a representation that is changing and moving. Formal notational symbols were organised imagistically, such as spiral formations or flashing numerals, with children visualizing number in a dynamic way. In the cross-sectional sample, 3% of the children displayed dynamic images of the number sequence whereas 10% of the high ability children used dynamic images.

Following this, an exploratory study of 77 high ability Grade 5 and 6 children investigated links between their understanding of the numeration system and their representations of the counting sequence 1-100 (Thomas & Mulligan, 1995). Analysis of children's explanations, and pictorial and notational recordings of the numbers 1-100 revealed three dimensions of external representation: (i) pictorial, ikonic, or notational characteristics, (ii) evidence of creative structural development of the number system, and, (iii) evidence for the static or dynamic nature of the internal representation. Children used a wide variety of internal images of which 30% used dynamic internal representations. Children with a high level of understanding of the numeration system showed evidence of both structure and dynamic imagery in their representations.

Other studies have also indicated that imagery plays a vital role in the development of conceptual understanding. Recent work (Brown & Presmeg, 1993) in which individual students' thinking was probed in clinical interviews indicated that students use imagery in the construction of mathematical meaning and involved very abstract and vague forms of imagery. Students with a greater relational understanding of mathematics tended to use more abstract forms of imagery such as dynamic and pattern imagery, while students with less relational understanding tended to rely on concrete and memory images.

We question then, how children's representations influence their conceptual understanding, and whether particular forms of imagery are more powerful than others in developing numerical processes. However, children's representations of numerical ideas cannot be investigated in isolation from the way children structure numerical relationships.

Structural Development of Number

Numeration involves the development of an increasingly sophisticated counting scheme and system of notation for recording the numbers generated. Initially a system of units, the counting scheme must drive the representation of assemblages partitioned into groupings of ten. The 'ten', while retaining its signification of ten units, becomes itself an iterable unit. This in turn may be operated on by the relation 'form a group of ten', to construct the new unit of 'hundred'. Arrays provide one conceptualisation of the multiplicative process, and can illustrate recursively the relations 'multiply by 10', 'multiply by 100', etc. in the numeration system.

Understanding number concepts and operations is based on the development of this structure as well as the development of a child's counting, grouping and notational capacities. For example, in multiplication and division concepts the development of a structure for forming equivalent groups and multiple counting strategies is crucial for conceptual understanding. The processes described here are not just verbal or notational; the role of imagery in representing these structures is essential.

A study by Mulligan (1996) examined how third graders spontaneously represented and calculated solutions to multiplicative word problems as they worked in a mini-classroom situation in which they were required to represent and explain their solution strategies. Children's internal conceptual structures of multiplication and division, described in earlier studies in terms of intuitive models (Mulligan & Mitchelmore, in press), were inferred from elements such as grouping, partitioning, counting and patterning of children's external representations, and explanations of these representations. Representations of multiplication strategies were classified as: (i) modeling the problem by *direct counting* one by one, without taking into account the structure of the equal sized groups; (ii) *repeated addition* of equivalent groups by rhythmic, skip counting or adding, and (iii) the *operation of multiplication,* represented as a binary operation. Intuitive strategies of division resembled those found for multiplication: (i) a rudimentary level, *direct counting,* where children shared and counted one by one; (ii) *repeated subtraction,* where the child started with the dividend and successively took away the numbers in each group, (iii) *repeated addition,* where the child started at 0 and built up to the dividend by counting or repeated adding, and (iv) the *operation* of multiplication as the inverse of division. Thus counting and calculation strategies wer also evidenced in children's drawn and written representations

Thus, our research now focuses on two aspects: types of imagery, and structural characteristics of number, in an analysis of children's representations of number. This research project seeks to ascertain whether the characteristics of children's representations are indicative of their level of conceptual understanding of number. More specifically, the study investigates the structure of children's internal representational systems for counting, grouping, regrouping, partitioning and estimation, the coherence and organisation of children's externally-produced representations, and their range of numerical understandings. This paper reports the developmental phase of the study. The research design and methodology, including interview tasks used during trialing, and preliminary findings will be presented for discussion and critical analysis.

Research Methodology

The research is an exploratory, descriptive study based on constructivist methodology widely used in contemporary mathematics education research. The qualitative analysis is highly descriptive but incorporates some quantitative measures of performance and coding of responses. Ten NSW Department of School Education primary schools from the metropolitan region of Sydney have been randomly selected for participation the study, from which a random sample of 120 Grade 2 students involving 12 students from each school (balanced for gender), have been chosen. Students will be interviewed individually, four times - once each in April and in November of 1996 and 1997, based on the design used in similar longitudinal studies (Carpenter & Moser, 1984; Mulligan, 1992). A pilot study was completed in March 1996 at a Sydney metropolitan primary school involving 10 Grade 2 children. The chief investigator and research assistant audiotaped and videotaped all interviews. Interview procedures and tasks were trialed to accommodate the range of abilities of the children and to allow sufficient time for children to draw representations.

Task-based interview procedures (Goldin, 1993) were used involving estimating, counting, grouping, regrouping and partitioning, subgrouping, splitting and other multiplicative tasks. Counting and grouping tasks focused on multiple and double counting strategies, counting as iterable units, partitioning and estimating (Mulligan, 1992; Steffe, 1991; Wright, 1992). Numeration tasks were based on structural elements of the number system, visualisation and number sense (Hiebert & Wearne, 1993; Thomas & Mulligan, 1995; Thomas, Mulligan & Goldin, 1994). Further questions focused on multiplicative structures such as arrays, combinatorial thinking, and splitting, were adapted from those used by Outhred (1993); Mulligan (1992), and Confrey (1994) respectively. Calculators will also be used in selected tasks. Some exemplars used during the trialing phase are presented in Table 1.

Analysis of Children's External Representations

Children will be asked to represent their solutions by modelling, drawing and symbolising their mental images. They will be asked to explain their representation and their solution process. The researcher will also probe for other available representations that the child can explain or represent. All interviews will be audiotaped and transcripts will be transcribed and coded for performance, three dimensions of representational characteristics, and for evidence of conceptual understanding. A selected sample of interviews will be videotaped. Results will be analysed for performance and solution processes initially for the whole sample to identify some general trends, and then according to individual performance and representational dimensions across the two-year period. Measures of association between the level of representation and the child's performance as an indicator of conceptual understanding will be identified and described. Profiles on each child will be compiled over the two-year period; analysis of the changes in the individual development of number concepts and strategies, and the corresponding representational capabilities will be analysed.

Table 1 Task-Based Interview Items

1. Numbers in Context
How old are you? What year is it? Close your eyes and tell me what you see. How old will you be in 2 years time? How old will you be in 5 years time?. What do you think the numbers mean. Close your eyes and tell me what you see when you give the answer. Draw and explain your response.
2. Counting sequence; visualisation tasks
Ask children to close their eyes and to imagine the numbers from 1 to 100, to draw, explain and write about what they see. Check for other sensory stimulation e.g. movement, position, colour, or feelings associated with numbers. Repeat the activity to find what the children imagine for numbers past 100; before 0; between 0 and 1; the smallest and largest number they can think of.
3. Numeration; grouping; rate
You can trade 2 small stickers for a large sticker. How many large stickers are worth the same as 4 small stickers? How many large stickers are worth the same as 2 large stickers and 4 small stickers?
4. Estimation; grouping; regrouping; addition. Show a collection of pencils in packets of 10 and as loose pencils. Ask, without counting, how many they think is in each packet? Why? Ask students to show 26 pencils. Use the pencils to make the number which is 9 larger than 26. Children who were successful with above tasks are asked if they can find another way to show 35.
5. Regrouping, addition -ones, tens, hundreds. Show the number 145 using bags, rolls and separate lollies. How many lollies will you have if you add 10 more to this number?
6. Structure; groups of groupings; determine pertinence of groups. Show a picture of (1) 35 (2) 80 (3) 140 dots randomly drawn. Ask child to guess how many dots there are drawn on this sheet of paper? Repeat for (1), (2) and (3). I am going to ask the same question to a friend who will be here after you. Could you do something so that, when I show them the sheet, they will be able to tell me very quickly how many dots there are? What would you do? Show the same picture but with groups of ten encircled with a black line and the group of ten tens encircled in red. Look what someone did for you. What do you think of it?
7. Estimation, addition: dot patterns and ten frame tasks.
Flash dot pattern for 8 and ask child to calculate number of dots. Reveal and ask child to count and add two dot cards together. Show ten frame for 7. How many more dots do we need to make 10? Show ten frame for 10 and for 6. How many more dots do we need to make 20?
8. Multiplicative tasks
(a) Splitting: If you had $24, how would you share it fairly between 2 children (same age). If 2 more children came along, how would you share it fairly with them. Repeat with $50 (only if $24 easily completed).
(b) Equal grouping: There are 2 tables in the room with 3 children seated at each table. How many children are there altogether (in the room)?
(d) Combinatorial: There are plain chips or salt and vinegar chips in small, medium or large packets. How many different packets of chips can you choose from?
(e) Quotition: There are 12 children and 3 children are seated at each table. How many tables are there?
(f) Array Structure: counting and grouping. Find total number in an array. Draw the array from memory. Show a 3 x 3 array of dots (stickers etc will suffice). Estimate. How many dots are there? Count the array. Find the total. Is there any other way of working out how many, more quickly?
(g) Sub-grouping. Provide model of cakes. Show how to share 2 cakes fairly between 4 people. How much do they get each?

Interview transcripts, together with the external pictorial and notational representations produced by the children will be coded and analysed. Intercoder reliability will be measured by an independent coder. The external representations will be considered with respect to three dimensions (based on operational definitions and coding procedures already developed in the pilot studies): (a) the type of representation (pictorial, ikonic and notational), from which we seek to infer characteristics of each child's internal imagistic representation; (b) the level of structural development of number concepts and processes evidenced in the representation; and (c) evidence of the static or dynamic nature of the image. For example, pictorial recordings will be defined as pictures drawn, or oral descriptions of objects given by a child. Ikonic recordings may include drawings of tally marks, squares, circles or dots that represent number. Notational recordings will be distinguished by the predominant use of numerals and symbols drawn in various formations such as an array.

The level of structural development of number will be interpreted from structural elements (i.e. counting, grouping, regrouping, partitioning and patterning) found in the recordings. For example, children showing evidence of a more developed multiplicative system may record a multiple counting sequence, and marks or pictures in a partial or complete ten by ten array structure. The static or dynamic nature of the image will be defined according to whether the recordings and children's explanations of their representations describe fixed or moving (changing) entities. Children's idiosyncratic representations of number will be explored fully in order to establish their influence on the child's general development of numerical strategies.

Implications

It is intended that the project will provide new knowledge about how children use imagery and structure in their pictorial, ikonic and notational representations of number. The analysis of representations will contribute to a new theoretical model for interpreting children's internal cognitive capabilities which will be particularly significant for assessing and assisting gifted and talented children, and those experiencing learning difficulties. The project will provide teachers with a practical range of innovative task-based assessment strategies and structured learning experiences that can be directly applied to improving mathematics teaching and learning, particularly for Early Numeracy Programs currently being trialed in Australia.

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