Young Students' Interpretations of Physical Angle Situations

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Each student in a sample of 144 Year 2 students was presented with models of three physical angle situations chosen from a set of nine. A bent straw was used as an abstract angle model. Students were questioned about how they would use the abstract angle model to represent each situation and what similarities they recognised between each pair of situations. The major factor influencing students' interpretations of the angle situations was the number of visible arms of the abstract angle. Implications are drawn for the teaching of angles.

Length and angle are arguably the two most basic concepts which we use to analyse our spatial environment, and they are also fundamental to the study of geometry. But, whereas most students master length early in primary school, angle causes difficulties well into secondary school (Mitchelmore, 1983). One difficulty is knowing what definition of angle to use. For some time now, mathematics educators have recommended treating angle in terms of turning, at least in the elementary stages. For example, our state mathematics syllabus for Years K-6 refers to angle as "the amount of turning between two lines about a common point" (New South Wales Department of Education, 1989, p. 79). It is argued that the development of the angle concept consists of establishing a particular relationship between the two arms of the angle; and a turn is a familiar, physical action which relates the two arms (Wilson & Adams, 1992).

A concept may be regarded as the end-product of a classification process which starts with the recognition of similarities between different experiences (Mitchelmore & White, 1995a). A definition of any given concept, then, draws out the predominant similarity in all the different situations in which it occurs. In the case of the New South Wales syllabus definition of angle, an amount of turning between two lines about a common point is regarded as encompassing the similarities in all situations involving angles. For example, the angle in a hill may be seen as the amount of turning between the imaginary horizontal and a line which gives the slope of the hill. It would therefore seem imperative to look at common physical experiences as a possible explanation for the complexities of the angle concept and for students' difficulties in learning it.

There has been very little research on how students actually conceptualise angles in their everyday experience. Davey & Pegg (1991) surveyed students from Grades 1 to 10 and reported that their descriptions of an angle went through a sequence of four stages: (1) a corner which is pointy or sharp; (2) a place where two lines meet; (3) the distance or area between two lines; and (4) the difference between the slopes of the lines. Clements & Battista (1989) found that Grade 3 students most commonly described an angle as a sloping line, a place where two lines meet, the two lines themselves, or a turn (the last only by students who had studied Logo). Such studies suggest that students do conceptualise their surroundings in terms of angles, and that the way they do this develops in sophistication as they get older. However, their knowledge of angle may be fragile and fragmented. For example, we have found (Mitchelmore & White, 1995b; White & Mitchelmore, 1995) that Year 4 students are often unaware of angle-related features linking different physical angle situations. Specific irrelevant features of the different situations appear to hinder abstraction of a general angle concept. In particular, it would seem that many students do not recognise turning in all contexts.

This paper reports the first part of a larger study exploring students' development of the concept of angle across Years 2 to 6. The specific research questions being investigated are as follows:

- What features influence Year 2 students' recognition of abstract angles in different physical situations?
- What angle-related similarities between different physical situations do Year 2 students recognise?

Method

Sample

The sample was gender-balanced and consisted of 144 Year 2 students from six schools in Sydney.

Materials

Based on earlier research (Mitchelmore, in press; Mitchelmore & White, 1995b, 1996; White & Mitchelmore , 1995), a series of nine physical angle situations were selected for study. There were four moveable situations (a small wheel, a door, a pair of scissors, and a Spanish [hand] fan) and five fixed situations (signposts, hills, road junctions, tiles and intersecting walls). (We reserve the terms "dynamic" and "static" to categorise angular similarities—see Table 3 below.) Each moveable situation was represented by a single physical model. Each of the fixed situations was represented by a set of three models representing a "neutral" configuration (angle 0° or 90°), an angle of 45°, and a "middle" angle (22.5° or 67.5°). Adjustable models of the fixed situations were specifically not used, in order to avoid suggesting a turning interpretation.

Twelve combinations of 3 physical situations were selected in which each of the 36 possible pairs occurred once and only once; each of the 9 situations occurred in 4 combinations. Each of the 12 combinations was then administered to 12 students. Thus, 48 students responded to each situation and 12 students responded to each pair of situations.

A drinking straw which could be bent at various angles was used as an abstract angle model. A second straw fixed at 45° was also used.

Interviews

A trained research assistant administered the interviews, which had been refined from the ones used in earlier work. For each situation, students were first asked to show how the abstract angle model could represent the given situation, and then to demonstrate an angle of 45° (shown on the abstract model) on the physical model. After they had responded to all three situations, they were asked what similarities they recognised between each pair. Probes were used at various points to clarify how students matched the abstract and the physical models.

Results

Matching abstract angles to physical situations

Students were asked to use the abstract angle model to show a general movement of the moveable situations or the 45° configuration of the fixed situations. The percentages of students who used the abstract model to represent the angular features of each situation in the standard manner are shown in Table 1. The term "standard manner" refers to the generally accepted angular modelling in which the vertex and the two arms of the abstract model match a point and two lines on the physical model. For the moveable situations, the vertex is the point of rotation and the opening of the arms represents the amount of rotation. For the fixed situations, the two arms represent two specific lines and the vertex is simply the point where these meet.

Table 1
Percentages of sample using abstract angle model to represent physical angles in standard manner

	General movement				45° configuration				
	Wheel	Door	Scissors	Fan	Signposts	Hills	Junctions	Tiles	Walls
Percentage standard	18	90	100	98	24	43	73	96	96

The only moveable situation to cause significant difficulties was the wheel. Very few students represented the turning motion of the wheel by an opening of the abstract angle model. All the remaining students (82%) made a rigid rotation of the angle model, either with the arms opened to a straight angle or with one arm bent out of the plane of rotation to represent the axis of rotation.

Among the fixed situations, the greatest difficulties occurred with the signposts and hills. In both these situations, students generally represented the sloping feature correctly but often did not represent a second (reference) line needed to make an angle. A 1-line method of representing slope was used by 72% of the students for the signpost and 53% for the hill. Even those students who did use a 2-line method tended not to represent the deviation from the neutral position, as most adults probably would. Thus, of those using a 2-line method, only 27% represented the signpost using a vertical reference line and only 44% used a horizontal reference line for the hill. Most of the students preferred to use as a reference line a line actually present in the physical model—the horizontal base of the signpost model or a vertical edge of the hill model.

Responses to the junctions situation showed many of the characteristics of responses to the signposts and hills situations, but to a lesser degree. Only 20% of the students used a 1-line method of representing road junctions.

Matching physical situations to abstract angles

Students were asked to set the moveable models, or to select the appropriate configuration of the fixed models, to represent a 45° angle shown on the abstract model. They were then asked to explain precisely how the arms and vertices of the abstract angle model were shown on the physical model. Students' responses are summarised in Table 2.

Table 2
Percentages of sample demonstrating and explaining abstract 45° angle on physical models

	Moveable situations				Fixed situations				
	Wheel	Door	Scissors	Fan	Signposts	Hills	Junctions	Tiles	Walls
Correct angle size	77	88	100	91	47	42	60	64	64
Standard explanation	21	44	28	17	43	45	44	100	91

Responses to the moveable situations were scored as the correct size if the angle was in the first quadrant. On this criterion only the wheel caused any problems, most of the incorrect responses falling in the range 90-180°. Responses to the fixed situations were scored as correct if the student selected the correct one of the three configurations. Almost all the errors consisted of choosing the middle configuration, very few selecting the neutral configuration. Students seemed sure that the 45° abstract model represented a non-neutral situation but were rather uncertain as to the deviation from the neutral position. Their uncertainty was most acute for the two slope contexts, signposts and hills.

The extent to which students used standard angle modelling to match the arms and vertex of the abstract angle to features of the physical model varied enormously. The easiest situations were the tiles and walls, where the two arms and the vertex are clear and unambiguous. The arms are also obvious for the scissors and the fan, but the vertex is not: Many students failed to place the vertex on the pivot. This nicety would probably fool many adults; in fact, 89% of the students placed the vertex on or near the pivot for the scissors and 79% for the fan. For junctions, many students placed the abstract angle model in the roads but did not align the two arms with the sides of the road (93% placed at least one arm parallel to one side of the road). For the remaining situations, many students could only match one arm of the abstract model to a feature of the physical model (the red line on the wheel, the top or bottom edge of the door, the sloping post, or the hillside). It was again noteworthy that those students who matched both arms of the abstract model to the signposts and hills models did not often use the neutral position as a reference: 70% related the signpost to the horizontal and 90% related the hill to the vertical.

Recognising angular similarities between physical situations

The similarities which students professed to recognise between two physical situations were classified as follows:

- Angle-related dynamic reasons, referring to the angular movement implicit in the two situations. Most such reasons referred to turning or opening, but some students could only say that the two situations showed a similar movement.
- Angle-related static reasons, referring to various features of the geometrical configuration implicit in the two situations. Two main categories of static reasons were found: (a) references to lines and points, such as "they both have two lines and a point" or "they both have angles", and (b) an explicit correspondence between the two configurations, such as demonstrating the match between the three tile corners and the slopes of the three signposts or stating that "the wheel can be turned to show the slope

of each hill". In both cases, these reasons were only accepted if students could demonstrate the correspondence correctly.

- Irrelevant similarities (such as "they are both made of wood"), inappropriate angle-related similarities (such as "the block on which the wheel is mounted has corners, like the walls") and appropriate angle-related similarities which could not be explained correctly.

Table 3 shows the number of students who recognised appropriate angle-related similarities between the various pairs of situations and correctly explained them, separated into dynamic reasons and static reasons. Only 5 students (scattered around the table) gave both dynamic and static reasons.

Table 3
Number of students recognising angle-related similarities between each pair of physical situations

	Wheel	Door	Scissors	Fan	Signposts	Hills	Junctions	Tiles	Walls
Wheel	-	6	5	6	0	0	2	1	0
Door	0	-	10	9	0	1	1	0	1
Scissors	0	0	-	9	1	3	0	3	2
Fan	1	0	2	-	2	1	2	0	2
Signposts	0	0	0	1	-	0	0	0	0
Hills	2	3	2	0	5	-	0	0	0
Junctions	3	3	5	3	6	5	-	0	0
Tiles	1	1	3	2	5	1	10	-	0
Walls	3	0	11	8	4	4	10	10	-

Note. The maximum in each cell was 12. Entries above the diagonal give the numbers of students recognising appropriate dynamic similarities. Entries below the diagonal give the numbers recognising and correctly explaining static similarities.

Dynamic reasons were never given to relate fixed situations and rarely to relate fixed and moveable situations. Dynamic reasons were much more often given to relate moveable situations. More than three quarters of the students related the scissors, fan and door dynamically, but only about half related these three situations dynamically to the wheel. Students almost always expressed dynamic similarities between the scissors, fan and door as "opening", whereas similarities between these three situations and the wheel were most often called "turning" and almost never "opening". It would appear that students in Year 2 regard the motion of a wheel as rather different from that of a fan, door or pair of scissors. The difference is consistent with the difficulty they have in identifying abstract angles in the wheel.

Static reasons were almost never given to relate moveable situations. Static reasons were predominantly given to relate fixed situations, but there was a considerable variation in their frequency. Almost all students related junctions, tiles and walls statically, but rather less than half related these three statically to signposts and hills. Static reasons were given more frequently than dynamic reasons to relate moveable and fixed situations, although the difference was not very great. (The large number of students who recognised a static similarity between walls and the scissors and fan is an anomaly which we cannot explain.)

Discussion

Many of the results from the present study (the frequency of 1-line representations of turns and slopes, common difficulties in representing the size of a rotation, and the relative ease of representing the abstract angles in the scissors and tiles situations) confirm the results of our earlier studies of Year 2 children. However, the wider selection of situations used in the present study makes it clearer that the major influence on angle conceptualisation is the number of arms of the abstract angle which are visibly present in the physical situation. The nine situations used may be classified according to the percentages of students who correctly matched the physical and abstract angles as follows.

- No line visible (wheel): 20%
- One line visible (door, signposts, hill): about 40%
- Two lines visible (scissors, fan, junctions, tiles, walls): allowing for some imprecision, between 80% and 100%

The significance of the number of visible lines is further shown by the percentages of students' responses showing the recognition of a similarity (either dynamic or static). The recognition rate was about 70% where both situations had two lines visible, compared to about 30% otherwise. The number of lines of the abstract angle which are physically visible seems to be far more important than whether the situation is moveable or fixed.

The salience of the lines visibly present in a situation is strikingly confirmed by some unintended results of changing the models used to represent the hill situation. In previous studies, we have used an adjustable hill model consisting of a sloping road hinged to a base. When this road was held up by a vertical support, many students used a vertical line as the second arm of the abstract angle. When the road was held up by hand, many represented the obtuse angle formed by the sloping road and the run-off road. In the models used in the present study, the hill was raised above the base so that there was no visible intersection between the sloping face and the base. But there were vertical supports, and students often used one of these for the second arm of the angle. Although even Year 2 students all seem to know that a horizontal hill does not slope, they show no tendency to measure the slope by its deviation from this neutral position. The same finding applies to the signpost situation.

Implications for teaching

The results of this study suggest that it should be possible to start teaching about angle as early as Year 2, provided the examples are limited to situations like the scissors, fan, junctions, tiles and walls models where both arms of the abstract angle are clearly present in the physical situation. Most Year 2 children seem to recognise angle-related similarities between such situations and to be able to represent the implicit angles by a simple abstract angle model. A study of these similarities and methods of representation could clear up their minor difficulties observed in this study (such as imprecision in representing the pivot of hinged objects and the arms of a road junction, and recognition of dynamic similarities between some of the fixed and moveable situations), as well as introducing the basic concept of angle. In this approach, angle would not be defined as "the amount of turning between two lines about a common point". If a definition must be given, perhaps "an opening between two lines" would be adequate for this stage.

The extension of the angle concept to situations where one or both arms are not visible should clearly be left until a later age. Before this can be done, students probably need to investigate turning and sloping as concepts in their own right. Experiences such as quantifying various amounts of turning and identifying the significance of the horizontal and vertical directions in space are likely to be especially valuable. Students could then be encouraged to identify the invisible lines which are needed to construct the abstract angle in turning and sloping situations, and hence to relate turns and slopes to other physical angle situations. It is difficult at present to predict a suitable age for such extension activities, but the replication of the present study in Years 4 and 6—completed since this paper was written—is likely to provide some answers.

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