What Does Mathematics Understanding Look Like?

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The concept of mathematical understanding is central to curriculum development, classroom interaction, and the training of mathematics teachers In this paper, some models of the growth of understanding that the literature presents are outlined Some results of a study that documented four primary teachers' mental models of, and beliefs about, different forms of understanding are reported It is proposed that linear models may restrict ways that these teachers plan lessons Questions for further research are raised

If you were asked to provide a metaphor for the development of understanding, how would you describe it? How does your model fit with your practices of curriculum planning or teaching and assessment of your own students? I wonder how many readers of this paper have considered such questions or discuss them with student teachers

There is widespread rhetoric about the development of "mathematical understanding", and many teachers, teacher educators, and curriculum documents stress that it is vital to develop it However, despite the fact that the meanings that people hold for the term mathematical understanding help shape teaching and teacher education, the varied meanings are rarely articulated This paper is an attempt to stimulate mathematics educators at all levels to address this anomaly

Given the complexity of understanding itself, any model or metaphor is bound to be inadequate (Pirie, 1988) However, it is useful to reflect on what people might mean when they use the term "mathematical understanding" as they expound its importance as an objective of school education One way to do this is to examine the models portrayed in mathematics education literature and to see if such models are also held by teachers In this paper, four teachers' mental models of mathematical understanding are described and tentative suggestions about how these may constrain planning and teaching are made

Models of Mathematical Understanding

The following summary of metaphors for mathematics understanding used by researchers is organised into three general categories: understanding as structured progress, understanding as forms of knowing, and understanding as process These are not the only way to group models of understanding but were "emergent categories" (after Lincoln & Guba, 1985) during a literature search in preparation for case studies of what teachers believe about mathematical understanding and what they do to develop it

This paper focuses mainly on models that were initiated in the 1960s—a period when how to develop students' understanding was a key focus of mathematics education research—with a resurgence of activity late in the 1980s In each category below, representative models are presented (See Mousley, 2003, for a fuller exposition)

Understanding as Structured Progress

My first group of models depicts the development of understanding as structural progression Use of "construction" models to describe the development of understanding followed a trend in sociology, but in mathematics education the notion was grounded mainly in psychology Terms such as "constructivism" clearly suggest a building process, but right from early development of such a notion, the idea of building from foundational understandings to higher levels of knowledge has not been as predictable as the metaphor suggests The process of fitting new knowledge with old has been portrayed as an active and interactive one For instance, Piaget described the development of understanding as a growing awareness of relationships, as inner experimentation, and as the internalisation of possible courses of action with specific purposes in mind—all activities involving sensory-motor activity aimed at the construction of objects (Piaget, 1950) Developing understanding involved increasing ability to hold several relationships in mind, permitting further abstraction and anticipation (Inhelder & Piaget, 1964)

Later, von Glasersfeld (1987) built on this work, portraying understanding as an organisational process and emphasising that cognitive activity is aimed at bringing about consistency:

> The experiencing organism now turns into a builder of cognitive structures, intending to solve such problems as the organism perceives or conceives … among which is the never ending problem of consistent organizations (of such structures) that we call understanding (p 7)

Sinclair (1987) also drew on Piaget's structural metaphor, noting that understanding of a mathematical concept is laboriously constructed over time He described how particular understandings serve as springboards for further learning, enabling progressive understanding to be built However, he noted that some understanding becomes backgrounded because there is a moment when a realisation becomes obvious, and the mind is released for other things The learner finds it hard to believe that there had been a time when a new idea was not present in the mind, so it can be difficult to go back to first principles Hence both children and inexpert teachers may find it difficult to explain the logical construction sequence that they themselves have used

This period also saw the emergence of Soviet psychology's model of developmental "zones" in western mathematics education theorising The contention is that teachers need to create learning situations that demand thinking, skills, and knowledge development that are just ahead their students' current zones of understanding Coming to grips with a concept sets up potential for movement into a further zone of development (e g Vygotsky, 1978)

Understanding as Forms of Knowing

Other researchers contrasted different forms of understanding An early example of this is Maslow (1966), who identified two different types of understanding The first was "scientific", where rational thought is reduced to lawful explanation The second was "suchness" understanding, which depends on contextual and qualitative experience, developing knowledge that can be referred beyond direct experience

Skemp (1976) acknowledged the work of Maslow and discussed his concept with Mellin-Olsen Skemp characterised two forms of understanding as different forms of knowing, claiming that these lead to two distinct kinds of mathematics He originally termed the two types of understanding "relational" and "instrumental" The former referred to the "building up a conceptual structure … from which its possessor can … produce an unlimited number of plans for getting from any starting point within his schema to any finishing point" (p 23) The latter involved learning by rote, but Skemp noted that, "for many pupils and their teachers the possession of … a rule, and ability to use it, was what they meant by 'understanding'" (p 20) He later identified two further forms of understanding: "logical" and "symbolic" (see Skemp, 1982)

Herscovics and Bergeron (1983) participated in debates of this time, later distinguishing four levels of understanding "Intuition" involves global perceptual awareness "Procedural" understanding involves a realisation of possibilities for transformation "Logico-physical" abstraction involves coming aware of physical invariants, of structure "Formalisation" involves generalisation to the use of mathematical procedures, realisation of possibilities for transfer to new contexts, and abstraction to the notational system However, they warned against thinking of these as sequential, noting that the child evolves simultaneously at different levels Gray and Tall (1994) suggested that such types of understanding are not easily placed on a linear framework because they interact They claimed that combinations of categories are needed to describe certain forms of cognitive development, particular understandings, behaviours, or outcomes of children's work Tall (1992) later suggested that a lattice should be used, with "concrete", "iconic", and "symbolic" forms of understanding laced against "relational", "instrumental" and "logical" forms

Sierpinska (1994) further modelled the process of mathematical understanding as a lattice, claiming that acts of understanding (e g explanations, validations) are interwoven with knowledge of particular situations (e g concepts, theories and problems) She distinguished between "acts of understanding" and "an understanding", with the latter being the potential to experience an act of understanding when necessary in specific contexts She distilled four categorisations of acts of understanding: "identification", "discrimination", "generalisation", and "synthesis"

Some researchers have categorised different forms of knowing hierarchically For example, van Heile and van Hiele-Geldof (1958) observed three forms of insight—pupils' understanding of what they are doing, of why they are doing it, and of when to do it They constructed a teaching sequence that can be used to move the students from very direct instruction to independent understanding, through "inquiry", "directed orientation", "explicating", and "explanation", to "free orientation" and finally "integration" Such work laid conceptual foundations for later hierarchical models such as the SOLO taxonomy (see, for example, Biggs & Collis, 1982)

Understanding as Process

Wittgenstein (1967) presented understanding as socio-linguistic activity He saw crucial connections between understanding and enculturation, claiming that people develop mathematical meaning through diverse "language games", where understanding depends on knowledge of conventional agreements He equated understanding with operating, acknowledging that a multiplicity of understandings relate to different types of operations

Pirie and Kieren (1989) also presented a grounded model They questioned the idea of categorising understanding because they believe that there is no such thing as understanding in the abstract For these researchers, mathematical understanding is described as "a process, grounded within a person, within a topic, within a particular environment" (p 39) They argued that understanding is a "whole, dynamic, levelled but non-linear, transcendentally recursive process" (Kieren & Pirie, 1991, p 78), "a continuing process of organising one's knowledge structures" (Pirie & Kieren, 1994, p 166) In this body of writing, Pirie and Kieren modelled mathematical understanding as a recursive phenomenon with thinking moving between levels of sophistication, each contained within succeeding levels Even when new concepts no longer draw on previous understandings, Pirie and Kieren identified a process of "folding back" to recapture what they call "inner-level knowing" (Kieren, 1990, p 197)

Methodology

In this paper I report on just a small part of case study research on teachers' views of "mathematical understanding", ways they think they develop it, and what they do in mathematics lessons to this end My focus here is on the results when four teachers who were asked to describe their own models of understanding

About 4 weeks were being spent in each of 4 classrooms in a rural school "David" and "Jan" were the teachers of the two Year 6 classes "Tracey" and "Robyn" taught the Year 2 classes All of these teachers were experienced practitioners, well respected by their colleagues as well as their students Each teacher was interviewed several times, and their mathematics lessons were videotaped The resulting audiotaped and videotaped data were analysed in order to find examples of what the teachers believed and did in relation to the development of their pupils' mathematical understanding The full report of this research (Mousley, 2003) is descriptive, with close reference to multimedia appendices

During interviews, these teachers were asked two questions relevant to the topic of this paper: (a) "If you were asked to describe children's development of mathematical understanding, what model would you use—maybe a metaphor or a picture that you imagine?" and (b) "Do you think there are different types of understanding?"

Metaphors for Understanding

When first asked about their images of mathematical understanding, three of the teachers described a spiral While Jan and Tracey described a simple linear model, Jan articulated this more clearly:

Jan: We use a spiral curriculum
Interviewer: What do you mean when you say that?
Jan: They learn to understand a series of things, and in the next grade they re-visit them and build on them It's a spiral, going up and up
Interviewer: Is that just your image? You said "We"
Jan: It's "we" It's a term we use often, and we plan that way It's the way that the school curriculum is set out
Interviewer: Is the CSF [curriculum document] structured that way?
Jan: Mainly It builds on each year, revising the topics Like groups, multiplying 2 numbers, then double digit by one, then long multiplication It keeps coming round
Interviewer: Do you imagine a single line?
Jan: I have not thought … probably … yes, I do It has to be because they are only learning one thing at a time
Interviewer: So is their understanding itself a single line—or the things they understand?
Jan: I am not sure what you mean (Paused) Understanding is the things they understand I see what you mean—like understanding itself No, they are the same They understanding a series of things so that is their understanding

Robyn described a detailed picture of a spiral of concepts and processes:

Robyn: I could not picture anything yesterday, but I thought about it It's like building blocks, where you stack one top of the other Not a straight line, (but) like a spiral because somehow ideas get repeated at the higher level, round and up (I asked for an example) I mean, they have to understand addition, not just be able to do it but to understand it before they can manage the next block, subtraction, but they also have to understand the addition when they do adding decimals later, and especially adding time—when it is not base ten So it's a stack of blocks, but you build on any idea revisiting it year after year It's colorful, really, and I think of it with colors, like red blocks for addition keep appearing above each other in the spiral

Interviewer: That sounds colorful So it's blocks, building blocks? Are they blocks of understanding?

Robyn: Yes, well blocks of things they can explain because they understand them

Interviewer: So they're concepts—maths ideas?

Robyn: Yes, and the processes, and time, and measurements, and fractions, and so on All they learn Things they can explain, ones they understand

David also used the spiral metaphor, but pictured more complexity:

David: … like a helix You know, DNA It's got lots of elements all connected at any level, but the levels develop in a long spiral

Interviewer: So is a level a grade level?

David: Perhaps, but not usually I think of them as smaller, much small, quite small; like decimals and percentages are linked on the same level, and next week they come together in money problems and that's all on one level

Interviewer: So it's quite a complex spiral—made of lots of linked ideas (David nodded) More like a network?

David: Yes, but in a spiral Do you know what I mean when I say a spiral curriculum?

Clearly the term "spiral curriculum" had influenced these teachers' perceptions of both teaching and learning in mathematics classrooms It is interesting that it was not used in any subject areas, and that the teachers were surprised when I pointed this out

Tracey: You are right Yes We don't have it Well, we do sometimes I can think of learning ball skills and athletics and things in phys ed, but you are right, we don't say that—spiral curriculum In maths it is obvious though because that is the way you teach it

There are 3 particularly interesting commonalities in what these teachers said about their images of spirals First, the teachers all described understanding as what is to be understood—in Robyn's words "things they can explain"—rather than a form of activity or an abstract notion in its own right

Second, their structural models fit with the term "constructivism" that was commonly used by them: e g , "We are constructivist teachers, mainly" (David) However, the metaphor portrayed a sense of teacher-designed sequences of new ideas and skills rather than the mental activity that authors such as Piaget and von Glasersfeld portrayed When I realised this I understood better the teachers' surprise at their own initial inability to talk about what they thought mathematical understanding is, because while one does not describe "understanding" as syllabus content, such content comprised the elements of the models they described in later interviews All four commented that they had used the term "mathematical understanding" many times, and believed that they taught "for understanding" but had not previously considered the phenomenon itself

Third, and more important, is the fact that the term "spiral curriculum", used in the school only in relation to mathematics content, seemed to have become unproblematic These teachers did not question the notion of the mathematics curriculum being fairly linear yet relatively repetitious from year to year This assumption influenced their planning and teaching as well as their expectations for student performance and future learning For example, when asked about a grade 6 girl who obviously had not understood what had just been taught in a lesson on multiplication, Jan declared, "… ideally she would know it all now, but we just have to accept the idea that she will need to come back to this work next year—and perhaps for several years"

Different Types of Understanding

When asked whether he knew about any different types of understanding, David responded immediately:

David	You mean, like abstract and concrete? … a lot of discussion that I had with [a Science lecturer] was about that
Interviewee:	What about any other types of understanding?
David	There was an article in *Common Denominator a* couple of months ago, about some others Relational was one I can't remember the others, but it struck me that it's important, relational I mean, understanding the connections between ideas

Jan also mentioned "abstract" and "concrete" understanding several times during our discussions, without prompting When I asked her if there seem to be different types of understanding, she asked me what I meant I suggested that she already knew about abstract and concrete understanding, and she said, "That covers it I think But then there's different types like understanding of graphs and equality and that sort of thing"

Robyn and Tracey frequently stressed the need for lots of experience with concrete materials before children can learn the abstract ideas I challenged this idea when talking to Tracey:

Interviewer:	Jamie-Lee talked about infinity this morning That's a really abstract idea
Tracey:	Yes, but a lot of them know it
Interviewer:	What do you think they know about it?
Tracey:	Just the idea of numbers going on forever and ever
Interviewer:	So you know how you said that they couldn't understand any abstract ideas without concrete experiences …
Tracey:	Well, usually, I mean I am thinking more about maths ideas when I say that
Interviewer:	Do you see abstract understanding as being quite different from concrete understanding Or are they just understandings about two different things
Tracey:	No, different types They can have one or the other, or both Say it's measurement They can understand how to measure centimetres, concrete, or what centimetres are as abstract but that is harder But if it is infinity like with Jamie-Lee that is all abstract, isn't it?
Interviewer:	Yes, it's a very important abstract idea Are there other sorts of understanding— other type besides concrete and abstract?
Tracey:	Possibly What are they?

The teachers' responses to my question about types of understanding were not unexpected Discussions about possible types of understanding are not common in teacher education courses let alone articles that teachers read It is interesting, though, that these results conflicted markedly with academics' responses during a pilot study for the research project When the first question above was asked of 11 mathematics education lecturers, three responded by saying "instrumental and relational", while a further eight people listed the following:

1	2	3	4
visual/spatial	instrumental	procedural	mastery
logical	relational	conceptual	ability to explain
numerical	logical		ability to use in context
inter-relational	symbolic		

5	6	7	8
practical	iconic	rote	thinking/explaining
theoretical	symbolic	concrete	modelling/applying
	relational	conceptual	patterning/generalising
	factual	abstract	abstracting
	conceptual		handing ambiguity
	analytical		

It was clear that the academics had a wide field of ideas to draw from in their reflection, discussion, and writing about mathematical understanding All four teachers' lessons demonstrated opportunities for children to develop many of these forms of understanding, so perhaps their not being to articulate different forms is not important to their effectiveness as mathematics teachers It was surprising, though—given that the term is used often in advice to mathematics teachers and trainees—that these teachers could not describe a range of types of understanding that children might exhibit

Again, the teachers' focus was on what is to be understood (e g , infinity) rather than different types of understanding that they could aim to develop

Conclusion

One cannot generalise from four teachers to mathematics teachers in general, but this component of the case study research suggests a need for further exploration Many research questions could be identified; e g , Is the spiral model very common across Victorian schools, and if so is it related to years of working with curriculum documents that present content in a particular mode? Most importantly, if the model is widespread, what effects does it seem to have on teachers' planning and expectations? If the metaphor is one that is held fairly commonly, does this change as the Victorian Curriculum and Assessment Authority's new framework of "essential learning" is introduced to all Victorian schools? Has a focus on the "new basics" in Queensland schools led to a comparatively wider variety of models of mathematical understanding in teachers' minds, and perhaps less structural ones?

In relation to forms of knowledge, we need to question whether, given that mathematical understanding is thought to be a key to children's success in our discipline, is it important that teacher educators stimulate their students to explore different meanings for this term Whether and how other models held by experienced teachers impact on their practices of planning, teaching and evaluating mathematics lessons would also be worthy of further study

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