Number Combinations and Arithmetic Structure: Implications for Early Algebra

Elizabeth Warren
Australian Catholic University
<E.Warren@mcauley.acu.edu.au>

This paper examines the influence certain number combinations have on children's ability to abstract arithmetic structure. The study replicates an investigation of Israeli and Canadian students, which found that certain number combinations significantly influenced the order in which operations are performed. Seventy-six Australian children participated in the study. The results indicated that, while the number combinations did not significantly influence the order in which the Australian children performed the operations, many exhibited misconceptions that reflected a very procedural approach to mathematics. Implications are drawn for the introduction of mental computation and early algebraic understanding in the early years of schooling.

Mathematics educators have long believed that algebra should precede arithmetic as it provides a foundation for algebra. In this world, school algebra is commonly perceived as derived from arithmetic. The perception of school algebra as generalised arithmetic is that algebraic structural rules emerge from rules that are valid in the world of whole numbers (Morris, 1999). In this instance, knowledge of mathematical structure is knowledge about the sets of mathematical objects, relationship between the objects and properties of these objects (Morris, 1999). It is about relationships between quantities (e.g., equivalence and inequality), properties of quantitative relationships (e.g., transitivity of equality), properties of operations (e.g., associativity and commutativity), and relationships between the operations (e.g., distributivity). In a beginning algebra course it is implicitly assumed that students are familiar with these concepts from their work with arithmetic. From repeated classroom experiences in arithmetic it is assumed that by inductive generalisation students arrive at an understanding of the structure of arithmetic.

The literature also suggests that many of the difficulties students experience with algebra seem to originate from their failure to understand the structure of arithmetic (e.g., Booth, 1988; Kieran, 1989; Warren, 2001). Linchevski and Linveh (2002) claimed that distilling this structure from the world of numbers is not as easy as it seems. They conjectured that certain number combinations often shift the focus of attention from the structure of the numerical properties of the given terms in such a way that wrong numerical values are assigned to expressions, "seducing naïve solvers into working in a given order of operations". For example, expressions such as $23 - 2 + 8$ encourage the students to perform addition before subtraction.

Juxtaposed against the distilling of structure from arithmetic experiences is the current emphasis on mental computation and number sense in many primary school curricular, with a focus on encouraging young children to recognize certain number combinations to assist them with mental computation (Morgan, 2000). Number sense involves using computational procedures flexibly (Markovitz & Sowder, 1994). In order to help young children develop this flexibility, attention is focused on particular number combinations involved in the calculation (Beishuizen, 1997). This emphasis has resulted in a move away from formal algorithms to an encouragement of allowing young children to invent and share their own strategies for solving arithmetic problems. This new emphasis is reflected in

Linchevski and Linveh (2002) suggest that this emphasis has often resulted in the bulk of teachers' and students' attention being focused on specific numbers used in expressions, and this pushes teachers to carefully choose specific numbers and operations that lead to more elegant and efficient calculations. For example, with expressions such as 31+4+6 the focus is initially on recognizing that 4+6 is 10 and then adding 10 to 31. While this understanding is important in developing numerical understanding, the impact this has on extracting arithmetic structure from numerical situations needs further investigation, as it is this extraction that is required to operate in an algebraic context.

Past research has shown that certain number combinations in fact bias the solution strategies that children chose to solve problems. For instance, Bell, Swan and Taylor (1981) showed that when children were given problems with the same structure but with different number combinations, they used different operations to solve the problems and the choice of the operations was dependent on the numbers given in the problem. More recently, other researchers investigating early arithmetic understanding have confirmed this tendency (e.g., Blote & Klein 2000; Heirdsfield, Cooper, Mulligan & Irons, 1999). Thus it is conjectured that numbers can create a' numerical' context just like words create a 'verbal' context and thus the numbers used can have a biasing effect on student's perceptions of arithmetic structure just like words can have a biasing effect on one's perceptions of sentences. This is particularly important as we move towards embedding algebraic reasoning in arithmetic reasoning, a move towards developing algebraic reasoning in parallel with number sense. Recent research has shown that young children are capable of thinking algebraically (Kaput & Blanton, 2001; Warren & Cooper, 2002). This is a shift *from* the traditional approach of algebraic reasoning that occurs after the development of arithmetic reasoning *to* algebraic reasoning that occurs in conjunction with arithmetic reasoning.

In order to test the conjecture that numbers can create their own numerical context, Linchevski and Linveh (2002) designed tasks that consisted of numerical expressions and presented these tasks to a sample of students from Israel and Canada. Algebraic structures were chosen that the children were familiar with and each structure was represented by three different numerical versions. In the first version the choice of numbers encouraged operating according to the correct algebraic structure. The second version the choice of numbers encouraged operating in a way that was opposed to the algebraic structure and the third version the combinations of numbers were such that they did not encourage the operator to act in any particular way. For example, for the structure a-bxc it was conjectured that 43-5x2 encouraged children to calculate 5x2 before performing the subtraction, and hence was classified as "going with" the structure. By contrast, 47-7x5 was considered to go against the structure as it was believed that children would tend to calculate 47-7 before carrying out the multiplication. Finally, 47-3x5 was considered to be neutral as it did not necessarily encourage children to go with the structure or to go against the structure. Their results indicated that certain number combinations did privilege differing orders in which the computation was carried out and thus interfered with the identification of the structure inherent in arithmetic expressions. In order to test this hypothesis in an Australian context two of the 9 items used in the Linchevski and Linveh study were chosen for investigation. The two items were seen as appropriate to the age of the Australian children participating in the study. Many of these children had had limited experience in dealing with expressions such as a-b+c+de.

Methodology

The sample comprised 76 children from four elementary schools in low to medium socio-economic areas in Australia. The children were all participants in a three-year longitudinal study investigating early literacy and numeracy development. By the conclusion of the study the children had completed Year 3, Year 4, and Year 5 of their elementary schooling. The average age of the sample at the beginning of the study was 8 years and 6 months and at the conclusion of the study was 10 years and 6 months. Prior to commencing the study, all had completed the first three years of formal education. In this particular instance only when the children were in Year 5 were they asked questions relating to the combination of operations. All had had some exposure to the order of operations.

With regard to the Israeli/Canadian study, two samples were used. One consisted of 59 sixth grade children (mean age of 11.5 years) and the other consisted of 78 seventh grade children (mean age of 12.5 years). In both instances, all the children had learnt the order operations prior to the data collection and they all had had plenty of opportunity to drill and discuss equivalent numerical expressions. Given the age of the children, it was decided to focus on the Year 6 children's responses in the data analysis as this was the sample that was more closely aligned with the Australian sample.

Instrument

For this particular study only two of the 9 items used in the Linchevski and Linveh study were chosen for investigation. The reason for this was that the children were younger and had had limited experience with dealing with expressions such as a-b+c+dxe. The items chosen were as follows:

Table 1
Items Chosen for the Written Test and Interview

Item		Examples used in the written tests		
		Goes with	Goes against	Neutral
1.	a-bxc	43-5x2	47-7x5	47-3x5
2.	a-b+c	28-8+5	27-7+3	28-5+3

The items were typed and printed with equal spaces between the operations and terms. This was done to avoid any visual cues that may influence children's choice of where to start. Kirshner (1989) suggested that such visual cues often led to misleading answers when dealing with algebraic expressions. For example, a–b x c could cue children to do the subtraction before performing the multiplication.

The items were incorporated into an extensive written test that included items related to ascertaining their solution strategies when finding unknowns. In order to minimize the influence each item had on the other, the items were randomly interspersed throughout the written test. Children were simply asked to evaluate the expressions. At the completion of the written test each child was interviewed. The interviews served to clarify the strategies children chose to solve the items. They were asked to explain how they had found their answer. Each interview was audio-taped for further data analysis.

In the case of the Israeli/Canadian children, the data was solely collected by using individual interviews.

Results

For the examples aligning with the algebraic expression a-bxc two approaches are possible. The first consists of performing multiplication before subtraction and second entails performing the subtraction before the multiplication. Similarly for the expression a-b+c there are also two strategies, doing the addition first and doing the subtraction first. The following table summarises the percent of children from the Australian sample and the Israeli/Canadian sample who used each of the strategies for each of these expressions.

Table 2

Percentage of the Sample Who Chose Each of the Strategies for Each Item.

Item	Particular examples and percentage responses					
	Goes with		Goes against		Neutral	
	Australia N=76	Israel Canada N=59	Australia	Israel Canada	Australia	Israel Canada
a-bxc	43-5x2		47-7x5		47-3x5	
Multiplication first	58%	61%	50%	33%	56%	52%
Subtraction first	25%	39%	35%	67%	22%	48%
a-b+c	28-8+5		27-7+3		28-5+3	
Addition first	39%	13%	43%	31%	48%	22%
Subtraction first	47%	87%	51%	69%	37%	78%

Significant differences were found between the Australian children and Israeli / Canadian children's responses to a-bxc ($\chi^2_5 = 31.578$, $p < .001$) and to a-b+c ($\chi^2_5 = 55.65$, $p < .001$). In order to explore the nature of these difference further Chi-squared tests were performed. In both instances children's responses to each of the examples (Goes with, Goes against and Neutral) were analysed. For the first item (a-bxc), there was no significant difference in the Australian children's responses to each of the examples ($\chi^2_2 = 3.662$, $p = .160$), indicating that in this instance the numbers chosen for the task were not influencing the order in which they performed the operations. In fact, the majority of the sample correctly performed the multiplication before the subtraction. By contrast, there was a significant difference in the responses given by the Israeli/Canadian children to the three examples ($\chi^2_2 = 16.358$, $p < .001$). The difference seemed to occur in their response to 47-7x5, with significantly more choosing subtraction for the first operations. This would indicate that for these children the numbers chosen for the expression did indeed influence the order in which they performed the operations.

For the item a-b+c, there was no significant difference for the Australian children responses the three examples ($\chi^2_2 = 2.755$, $p = 252$). A similar trend existed in the

the numbers chosen to represent the expression did not significantly influence the order in which they performed the operations. It seems that the significant difference that existed between the responses of the two samples was related to the order in which they performed the operations. The Australian children tended to be fairly evenly split between those who chose addition before subtraction and vice versa, suggesting that many did not understand the ordered conventions, or other learning was interfering with their choices. By contrast, the majority of the Israeli/Canadian children (up to 87%) chose to perform subtraction before addition indicating that most could apply the correct order of operations to their calculations. The responses to the interviews begin to provide insights into the trends in the Australian children's responses.

Interview Responses

In the interview, students were presented with their responses to the written test and were asked to explain how they worked out the answer and why they had taken that approach. Commonly children provided utterances that simply followed the order in which the operations were carried out. Some typical answers were:

Utterances that mirrored their solutions.

Well I timesed those two together and that equals 10 and then you have to take away from 43 and that is 33.

I got 47 take 7 and then I times that answer by 5 and I got 200 as my answer

Many children followed the procedures that they used for obtaining the answers without asking the question of whether they were right or wrong. For example:

43-5x2
I did 43 take 5 and then I timesed by 2 and I got 48

28-5+3
Well I did 5 plus 3 is 8 and then I did 27 take 8 and that is 21. *(the answer on the written test was 21).*

It seems that in these instances talking aloud did not challenge their erroneous thinking. They simply ensured that their utterances matched their solutions without asking whether their solutions were correct or incorrect.

The responses to the interview also served to provide insights into why particular strategies, such as addition preceding subtraction in the second example, were adopted. One child consistently separated each examples into two discrete operations and then combined the answers for each.

27-2+3
I done 27 take 7 that is 20 and then 7 plus 3 is 10 and then add the together I got 30. *(the answer on the written test was 30)*

47-3x5
47 take 3 is 44 and 3 times 5 is 15 and then added them together [44 plus 15] and that is 59

An analysis of the Australian children's responses to the written test indicated that seventeen children (22%) consistently processed all the examples from left to right, twenty eight children (37%) consistently did the last part first and only two children correctly answered all the problems. For the rest of the sample the pattern of response varied. The next section gives some reasons for why they chose a particular strategy. In each instance the following code was used (I = interviewer, C1 = Child 1, C2 = Child 2; C3 = Child 3…).

Consistently calculating the last operation first (37%).

I: How did you do this one? –[Pointing to 47-7x5]

C1: 7 times 5 is 35 and then I went 47 take 35.

I: Why did you go to the end part of the sum first to work out all of those?

C1: Because it would have been easier. Cause if I just writed that I would be too difficult.

I: How did you know to do the multiplication first before doing the subtraction.

C1: Because I just I thought about it and I thought it would be easier

This thinking was then generalised across all operations.

C2: You always do the second part first because it is easier. It is easier to go 8+5 is 13 and then take this from 28.

C3: You do the last part first because it is easier… the numbers are smaller.

Consistently calculating from left to right (22%)

C4: So doesn't really matter what would goes first. You just start with the start of the sum and work across So 28 minus 5 is 23 plus.. It should be 26

C5: You just start at the beginning – they are easy I can just do them.

Some children had heard of BOMDAS in class. Two used the rule successfully whereas others were confused by the rule.

Performing a teacher given rule.

C6: Well first BOMDAS I thought of that It's like this word and each letter Like b is brackets, a is addition, m is multiplication minus ah is s for minus? and that's all I know

C7: There was a rule for this and there would be times and divide – you have to use those first and plus and minus second. Then there's an odd thing but I don't get this thing.

C8: I remembered the word BOMDAS and multiplication comes first so said 2 times 5 which is 10 and 43 take away 10 is 33. I did the same. Addition comes first so I said 5 plus 3 equals 8 take away 28 which is 20

The response given by C7 suggests that the introduction of BOMDAS may have also influenced the choice of addition before subtraction as 'a' comes before 's' in the acronym

Discussion and Conclusions

For the Australian children the numbers did not seem to influence the order in which they carried out operations. They seemed more procedural in their approach either always working from left to right (22%) or consistently performing the last part first (37%). Performing the last part first for them resulted in smaller numbers that they had to compute. As one said, *you do multiplication first because it is bigger numbers so if you do it first it is easier* and *it is easier to add in my head than to take away so you do that first*. Interestingly for these children there seemed to be a shift from working from left to right to working on the last part first. The reasons given for this seemed to imply that these children were taking this option in order to reduce their cognitive load, that is, the size of the numbers with which they had to deal with. It could be conjectured that intuitively they knew that multiplication resulted in larger numbers and thus it was easier to perform multiplication before subtraction, that is the last part first. They then seemed to generalise this thinking for all the examples. How this decision is influenced by the fact the first number was always larger than the following two (e.g., 47 compared with 7 and 5) needs further research as it may be that not only do the particular numbers chosen for the problem influence the number context but also the magnitude of the numbers used in the problem have a biasing effect on arithmetic structure.

Fischbein (1999) begins to give some insights into the trends in these children's responses. It seems that many of these children have false intuitive understandings that influence the order in which to perform the operations. He makes the distinction between procedural schemata and structural schemata. In this instance "A schema is a program which enables the individual to record, process, control and mentally integrate informations, and to react meaningfully and efficiently to the environmental stimuli" (Fischbein, 1999). Procedural or Action Schema are content bound and influenced by ones intuitions (e.g., multiplication results in bigger numbers, it is easier to add than subtract, always start at the beginning) and the context of the problem (e.g., the types of numbers chosen). Often students find themselves in conflict with formal mathematics that is based on definitions and deductively based theorems and their intuitive understanding (cognition) of the situation. The effect is that the student does not accept and does not understand the formal definition or statement, or in many instances even when he/she seems to initially understand he/she tends to forget the definition and revert to intuitive interpretations. An example of this is their understanding of their teacher's explanation of how to decide the order in which to perform operations. For example, Ben said *There was a rule for this and there would be times and divide – you have to use those first and plus and minus second. Then there's an odd thing but I don't get this thing.* Ben then proceeded to work from left to right for all the problems.

Unlike the Israeli/Canadian children, the Queensland children have only recently had an emphasis on number sense embedded in the new curriculum. Previously the emphasis has been on algorithmic thinking. This lack of emphasis could explain the differences in the trends in responses. It may be that as we move to a focus on number sense that indeed the numbers we choose to make explicit the inherent patterns in numbers may influence the trends in their responses. The Israeli/Canadian study suggests that numerical contexts need to be taken into consideration when endeavouring to abstract the arithmetic structure.

Exploring numbers for the purpose of developing number sense and the exploring numbers for the purpose of early algebraic understanding may indeed be in conflict with each other. It may be that in order to cater for this conflict that the abstraction of arithmetical structure needs to occur in other worlds besides the world of whole numbers, that is, developing reasoning where the objects of reasoning are the relationships between unmeasured quantities (Davydov, 1982). This involves the development of children's ability to conceptualise and reason about structural properties of operations and relations in the context of unmeasured physical quantities and then linking this reasoning to numerical operations, It is conjectured that this approach limits the influence numbers have on abstracting the underlying structures of arithmetic. It is also conjectured that children generalise in a way that they are initially taught and that this can result in the construction of schemata at an early age that have a strong inherent robustness. Linchevski and Linveh (2002) claim that in many instances these 'old' schemata become tacit models of comprehension. Thus even after repeated instruction and cognitive challenges, initial understandings persist.

Teaching Implication

As we chose specific numbers to assist in developing an understanding of number sense we must be aware that these may indeed work against abstracting the structure of arithmetic. Thus we need to consider exploring these combinations in a variety of number contexts to ensure that the propensity to combine these combinations is considered within the context in which they are situated.

References

Beishuizen, M. (1997). Mental arithmetic and strategy use with indirect number problems to one hundred. *Learning & Instruction, 7*(1), 87–106.

Bell, A., & Swan, P. & Taylor, G. (1984). Choice of operation in verbal arithmetic problems: The effects of number size, problem structure and context. *Educational Studies in Mathematics*, *15*(2), 129–147.

Blanton, M., & Kaput, J. (2001). Algebrafying the elementary mathematics experiences. In H. Chick (Ed.), *Algebra*. Melbourne: ICMI.

Blote, A. W., & Klein, A. S. (2000). Mental computation and conceptual understanding. *Learning & Instruction, 10*(3), 221–247.

Booth, L. (1988). Children's difficulties in beginning algebra. In A. F. Coxford (Ed.), *The ideas of algebra. K-12.* Reston, VA: National Council of Teacher of Mathematics, 20–32.

Davydov, V. V. (1982). The psychological characteristics of the formation of elementary mathematical operations in children. In T. P. Carpenter, J. Moser, & T. Romberg, A. (Eds.), *Addition and subtraction: A cognitive perspective*. (pp. 224–239). Hillsdale, NJ: Lawrence Erlbaum Associates.

Fischbein, E. (1999). Intuitions and schemata in mathematical reasoning. *Educational Studies in Mathematics, 38,* 11–50.

Heirdsfield, A., Cooper, T. J., Mulligan, J., & Irons, C. J. (1999). Children's mental multiplication and division strategies. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 3, pages 89–96. Haifa, Israel: Israel Institute of Technology.

Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wagner and C. Kieren (Eds,). (pp. 33–56). National Council of Teachers of Mathematics.

Kirshner, D. (1989). The visual syntax of algebra. *Journal for Research in Mathematics Education, 20,* 274–287.

Linchevski, L., & Linveh, D. (2002). The competition between numbers and structure: why expressions with identical algebraic structures trigger different interpretations. *Focus on Learning Problems in Mathematics, 24*(2), 20–37.

Markovitz, Z., & Sowder, J. (1994). Developing number sense: An intervention study in grade 7. *Journal For Research in Mathematics Education 25*(1), 4–30.

Morgan, G. (2000). Put mental computation first? *Australian Primary Mathematics Classroom. 5*(3), 4–9.

Queensland Studies Authority (2003). Draft mathematics syllabus. Accessed March 2003 from http://www.qsa.qld.edu.au.

Warren, E., & Cooper T. (2002). Arithmetic and quasi-variables: A Year 2 lesson to introduce algebra in the early years. Mathematics Education Group of Australasia.

Warren, E. (2001). Algebraic understanding and the importance of operation sense. In M. van den Heuvel-Penhuizen (Ed.), *Proceedings of the 25th conference of the international group for the Psychology of Mathematics Education, 4,* 399–406. Utrech, Netherlands: PME.