Interactive Animation Provides a Vehicle for Exploring Students' Understandings of Derivatives

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Animated graphics can provide a basis for discussion of calculus concepts. This paper reports analysis of the responses from nineteen undergraduate mathematics students who were asked to write a work sheet based on a simple animation linking the graph of a function with its gradient function. Their responses were benchmarked against those of experienced teachers. This open assessment task provided valuable insights into students' perceptions of important aspects of both the applet and the derivative concept.

Computer animations are now an expected part of online or CD-ROM based teaching materials. As mathematics teachers, we may enjoy exploring these illustrations and admire the creativity of the author in providing an innovative representation that emphasises key features of a familiar concept. Do our students, who may have gaps in conceptual understanding, notice the range of features evident to the experienced teacher? Students' responses to such interactive graphics may suggest areas of their conceptual understanding that need further attention in teaching.

This paper reports the results of an exercise, completed by three mathematics staff and nineteen students, which required them to write material focused on an interactive graphical animation (a simple JAVA applet) called surfing derivatives shown in figure 1. The aim of this exercise was to force students to clarify and refine their own ideas about functions and their derivatives, in order to put them into words. The students' responses to this illustration give insight into their schema for understanding derivatives.

Background

Many authors have commented positively on the place of multimedia and computer animations in the teaching of calculus concepts. For recent examples see Abu, Hassan and Sahib (2001), Klotz (2002), or Giraldo (2002). In particular, Malabar and Poutney (2002) commented that working with computer visualisation assisted the construction of mathematical understanding through provision of new and vivid experiences. The activities used in their study prompted students to verbalise their understandings of the concepts displayed. This invoked higher order thinking in the forms that Bloom (1956) termed focusing on information transfer, interpreting, implications and evaluation. In teaching and assessing mathematics is it relatively common to set exercises that require recall and the use of practiced routines, but it is harder to create tasks which require students to discuss or write about their understanding. Interactive applets may be a springboard for such tasks.

Evidence that interactive animations can be used as a stimulus to promote higher order thinking suggests that they could have a valuable place as a focus for diagnostic or assessment tasks as well as in teaching activities. However few studies report the use of multimedia animations in assessing students' conceptual understandings. In one such study, Hare (1997) videotaped and analysed the conversations of students working collaboratively in an interactive environment. In order to verbalise their understanding, students were forced to clarify their own ideas. This revealed weaknesses in students' conceptual understandings of calculus, compounded by their incorrect use of language.

This approach, while useful in research, is not practical as a regular classroom practice. Thus in the study reported in the present paper, a logistically simpler exercise, featuring an interactive graphical animation, was trialed. Students were required to verbalise their understanding through a written task that aimed to explore their conceptual understanding of differentiation.

This Study

This study was planned to give some insights into the individual nature of user responses triggered by the experience of exploring a simple computer animation. For this purpose an exercise was set which required each participant to write material based on their individual understanding of derivatives and their response to a particular interactive animated illustration. The resulting scripts were analysed.

Participants and Task

The exercise discussed in this paper was completed by nineteen undergraduate students who were studying an introductory course on functions and calculus, and also by three tertiary mathematics teachers who were not currently teaching this group of students. The task formed part of an assessed assignment for the students. This work was due for submission at the end of week ten of a thirteen-week lecture and tutorial series. The students had met the concept of derivatives in their secondary school mathematics and had spent some weeks studying differentiation and integration in this course, which took a multi-representational approach to functions and calculus, with the support of CAS calculators.

The participants were directed to the URL for the surfing derivative animation, shown in Figure 1. This applet was chosen because it was simple and appealing, it linked closely to graphical work done in the course, and it illustrated a foundational concept.

The participants were asked to write a guide for this illustration. The guide was to be in the form a worksheet suitable for someone who was new to calculus. The directions went on to say that this guide should explain the concept and how the applet illustrated this, tell the novice calculus student what to look for and suggest options for exploring the concept. The instructions also said that the worksheet should include questions that test understanding. Part two of the set task required participants to provide answers to these questions.

A worksheet was chosen as the context for the students' writing because most of students in this group were pre-service teachers. Although the task was deliberately very open, a sample worksheet, suitable for use with an interactive animation illustrating the sum of an arithmetic progression, was provided as an example of suitable length, depth of analysis and style.

The Animation

The interactive animation *surfing derivatives* is set entirely within the graphical representation of a function and its derivative. The graph of a function is shown and a red dot represents a point on this curve. A short line segment through this dot is used to represent the tangent to the curve at that point. The user may *drag* the red point and hence view the slope of the tangent line at any point along the curve. A triangle of base one is subtended from the tangent line so that the height of that triangle corresponds to the gradient or derivative of the curve at each point. Clicking on the *Trace* button activates the lower section on the applet. Here the gradient of the initial function, at many points, is plotted around a horizontal axis as the user drags the red point along the curve. This builds an image of the gradient function. One option provided by this applet is a *Man* button. When this is clicked *on*, a diagrammatic man is shown standing on the line segment of the tangent. He is referred to as a surfer, the curve as a wave and the line segment as the surfboard.

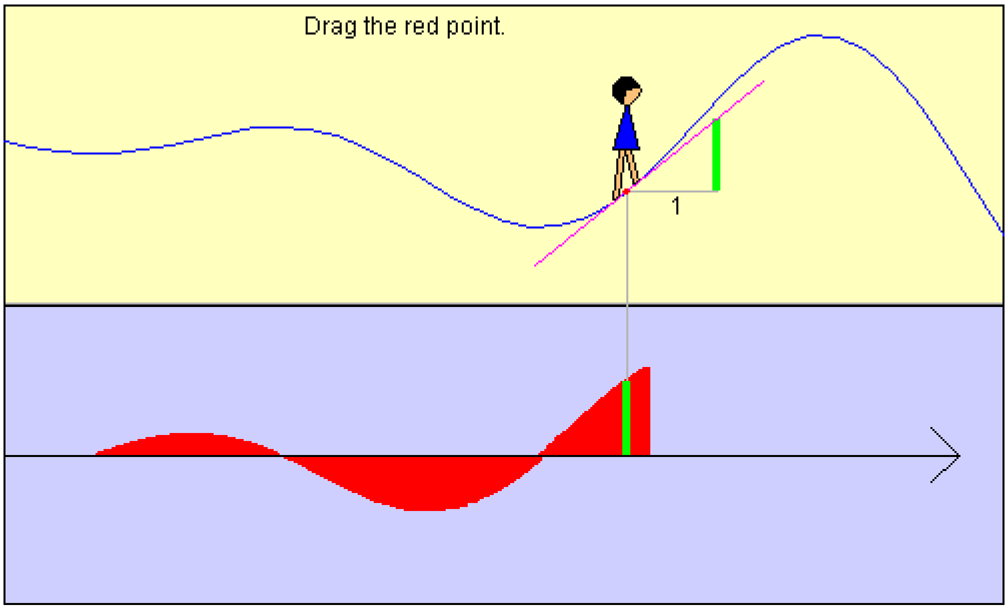

Figure 1. Screen dump from surfing derivatives animation.
(http://www.ies.co.jp/math/java/calc/doukan/doukan.html, retrieved 24/03/03_)

The Analysis

The researchers' intention was that interaction with the applet would act as a catalyst encouraging students to articulate their understandings of the foundational concept of the derivative as a rate of change. The responses of the experienced teachers were used to provide a benchmark for evaluating the comprehensiveness and quality of the student worksheets. The analysis of students work, discussed below, considered: use of the model, understanding of the concept of slope of tangent as derivative at a point, use of appropriate mathematical language, and the representational focus of questions set as either algebraic or graphical.

Results and Discussion

In their various worksheets many features of the applet and the understanding of derivative that it reinforces were addressed by both the teachers and the students. However many students missed some possible features. Perhaps not surprisingly, only one student and the experienced teachers invited criticism of the applet. We will now look at these features in some detail.

Table 1, below, provides a summary of the features that the experienced teachers addressed, along with a list of the students who also chose to include these features in their worksheet guide for a novice student. A list of identification codes for the students who chose to focus on algebra is also included. None of the three teachers took this approach.

Table 1

Features of the Applet or of Derivatives Addressed by Students in their Worksheets.

Teacher benchmark: Features of worksheet	*Students who incorporated this feature in their worksheet*
Mathematics:	
Defined derivative graphically (rate of change, slope, gradient)	1,2,3,7,8,9,11,12,13,14,16,17,18,19
Used conventional mathematical language	2,4,5,6,7,8,9,10,11,12,13,15,16,17,19
Slope of tangent- derivative at a point	3,4,11
Drew attention to sections of positive, negative and zero gradient for *f(x)*	2,3,4,5,6,7,11,13,14,15,17
Drew attention to maximum and minimum values of *f(x)*.	1,6,8,13,15
Linked key details of $f'(x)$ to key features of *f(x)*	1,8,12,13,17
Emphasised rate or change equals rise over run.	2,3,14,15,19
Applet:	4,5,7,9,11,14,16,17
Indicated that surf board represents tangent	
Drew attention to details of applet, eg, green line linking f(x) with $f'(x)$	3,4,8,9,12,15
Criticised the applet	14
Students only:	
Focused on algebraic representation of derivative.	5,6,7,11,14,16

Benchmark: The Teachers' Responses

The teachers, whose work was used as a benchmark, each defined the notion of derivative in terms of rate of change and linked this to the slope of a curve. They explained that the slope of a curve at a point is the same as the slope of the line that is a tangent to the curve at that point. The teachers pointed out that the surfboard represents the tangent, and then examined the slope of the board at various points along the wave. They pointed out the links between the function and the derivative curve, and vice versa, at features such as maxima, minima, and sections of positive and negative slope.

Student Responses

It is interesting to note not so much what students included but the number of students who did not include many possible key features of derivatives illustrated by this applet. For example, eleven students noted when the gradient of the tangent, represented by the surfboard, was positive, negative or zero but only five students explicitly drew attention to maximum and minimum values. Only students 8, 13 and 15 referred to positive, negative and zero gradients as well as maximum and minimum points.

While most students defined the derivative in terms of rate of change only three made reference to rate of change at a single point. Most students did not explicitly draw attention to the various features of the applet and the mathematics that they depicted. Only eight students stated that the surfboard represented the tangent or gradient of the curve but all except three invited the novice student to consider the slope of the board. Only six students drew attention to the triangle and line linking the two curves and fewer pointed out the triangle's meaning in terms of calculating the derivative at a point in a graphical context. Many referred to sections of positive, negative and zero gradients, but had difficulty articulating this using standard mathematical language such as maximum or minimum. Students 3 and 10's questions and suggested answers reflect this difficulty.

> Student 3: Drag the red point along the curve. What do you notice about the slope of the surfboard?
>
> *Answer: It moves up and down according to the wave.*
>
> Student 10:Why is the derivative function 0 when the surfer is on top of the wave?
>
> *Answer: This is because at the top of the wave the gradient is zero.*

Student 17, on the other hand, was able to articulate a clear understanding of the image. An excerpt from his worksheet, below, demonstrates this strength. Student 17 wrote:

> Read the information given. What is the function f(x) represented by?
>
> *Answer: The curve of the wave.*
>
> Look at the position of the man and the board. What is the significance of the board? Explain.
>
> *Answer: The board is like a tangent line to the curve. The slope, or angle, of the board is relative to the derivative of the function at that point.*
>
> Initially the board is flat, what does this mean about the derivative of f(x)?
>
> *Answer: The slope of the curve must be equal to zero. In terms of f(x) , this means that the function is neither increasing or decreasing.*
>
> Click on the trace option and slowly drag the red dot across the wave. What does the trace represent?
>
> *Answer: The derivative of f(x)*
>
> Drag the surfer to the top of the first crest. Predict what will happen and why to the trace, as you move the surfer past this crest.
>
> *Answer: The trace will become negative, because the derivative becomes negative, ie. The function is decreasing with respect to x.*

Limitations of the Applet

The teachers raised two criticisms of the applet. First, when the trace facility was activated the slope of the curve at each point was transferred to the axes below the original function. This built up a shaded curve representing the derivative function. The teachers were concerned that illustrating the gradient function by a shaded curve may cause confusion with Riemann sum techniques for finding area under a curve. There was no evidence that these students found that aspect of the illustration confusing. Only one student mentioned the build up of the shaded curve and the different images given for slow and fast dragging, even though by the time the assignment was due in these students had studied both anti-differentiation and area under curves. Student 13 pointed this out but did not perceive any problem:

> Drag the surfer across the waves quickly. What do you notice?
>
> *Answer: Only some of the derivative is drawn, it looks like a bar graph.*
>
> Now drag him slowly and describe what happens. Why do you think there is a difference between the two speeds?
>
> *Answer: The graph is drawn continuously; therefore it is all coloured in. This gives an accurate representation of the rate of change of the slope.*

One teacher raised this second criticism:

> I feel that the example of a surfboard has its problems in that no real life surfer would last long with significant portions of the board under water. Also, surfers tend to have little success surfing up hill as does this surfer.

No student made any comment on the suitability of the implied model but, in unthinkingly accepting the image, many students refer to the surfer going up the waves and some students produced tangled language. For example:

> Student 6 wrote: Waves that have low curves are called minimums and waves that have high curves can be called maximums.
>
> Student 2 wrote: The surf board slopes up and down according to the wave's ascent or descent…When the surfboard slopes upwards from left to right this mean the wave's rate of change is positive (increasing). [No variables had been defined, Student 2 only stated that a derivative $f'(x)$, is the rate of change of a given function *f(x)*.]

One student, number 14, criticised both the lack of an algebraic definition for the function and the lack of x- or y-coordinates marked on the graph, saying this made it very difficult to calculate the gradient. This criticism illustrates her inability to deal with the concept at a general rather than specific level.

A further limitation, possibly due to a decision to simplify the presentation, required an extra level of understanding. By representing the slope of the tangent using a triangle with base one unit, students could confuse the value of the derivative with the height of the triangle, rather than the ratio of rise to run. For example, Student 9 wrote

> The gradient of the surfboard is equal to the green line because this is the rise when run is equal to one…What does the green line represent in both the applet and the diagram?
>
> *Answer: The green line represents the gradient of the tangent or surfboard.*

Focus on the Algebraic Representation of Derivative

A particularly interesting feature of seven student's work was their emphasis on the algebraic, symbolic representation of derivative despite the graphical representation in this exercise. It was clear that despite the graphical work, which had been undertaken in their course of study, what they saw as the important message to convey about derivatives to a novice student was the algebraic rule for the derivative of a polynomial. In a similar way to students described by Pierce and Stacey (2002) as having a negative attitude towards the use of Computer Algebra Systems, these students seem to place a greater value on knowing the rule for a routine procedure than on visual representation and conceptual understanding. Perhaps this had been their own point of entry to this area of mathematics at school. Student 14, for example, started her worksheet this way:

> The derivative function produces the slope (gradient) of the curve represented by y=f(x) at any value of x. the derivative function is denoted by f’, and the process of finding the derivative is called differentiation. To find the derivative of a function, you multiply down by the power and then reduce the power by one.

This was followed by several algebraic examples before there was any reference at all to the applet.

In his introduction Student 19 provides his novice student with:

> Formula: f’(x)=lim$_{_x_0}$f(x+_x)-f(x)/ _x, *provided the limit exists*

Student 19 then sends the novice to the applet and instructs them to:

> Play around with the surfer and get a feel of how the slope varies for different sections of the curve…Now back to the formula.

When students wrote about their mathematical understandings, their own deeply held conceptions and misconceptions were highlighted. This concurs with the experience reported by Bezuidenhout (1998) who also used computer animations but found that the initially constructed schema can impede the development of more sophisticated understandings of fundamental concepts.

Conclusions

The exercise reported in this study required students to verbalise their perception of the key features of an animated representation of the concept of the derivative as the gradient of a curve. This proved to be a successful task for prompting higher order thinking as it assessed students’ ability to articulate their understanding of derivatives, applied in a new context. The students’ written responses provided the teacher with valuable information on which to base follow-up teaching.

The better students were able to use the animation to discuss the link from a function to its derivative and vice versa. They could explain the representation of the tangent as a surfboard, and provide illustrative examples. A second group of students was able to correctly draw attention to some important features of the applet but gave incomplete responses. On the other hand, despite the use of multiple representations throughout their recent course, the study also showed that some students preferred algebra and had strongly associated the concept of the derivative with an algebraic routine for polynomial functions. Even though the task was to use a graphical representation to explain the concept, they chose to focus on the algebra.

Encouraging students, undertaking a written task, to verbalise their understanding of the mathematics represented by simple interactive computer applet, proved to be a valuable way to elicit information on their own internal schema, understanding of concepts and any significant biases or deficiencies.

References

Abu, N. A., Hassan, M. D. & Sahib, S. (2001). Mathematical animations: the art of teaching. *31st ASEE/IEEE Frontiers in Education Conference SIC-10.Reno,NV.*

Bezuidenhout, J. (1998). First-year university students’ understanding of rate of change. *International Journal of Mathematics Education in Science and Technology, 29*(3) 389–399.

Bloom, B. S. (Ed.)(1956). Taxonomy of educational objectives: The classification of educational goals: Handbook I, Cognitive domain. Toronto: Longmans, Green.

Giraldo, J. (2002). Using animations in the teaching of calculus concepts. *Society for Information Technology and Teacher Education International Conference 2002(1),* 1078–1079. [Online]

Hare A. C. (1997) An investigation of the behaviours of calculus students working collaboratively in an interactive software environment. *Disserstion Abstracts International Section A: Humanities and Social Sciences. 57(9-A)* 3862. US: University Microfilms International.

Klotz, E. (2002). Teaching calculus with digital libraries. On: I. Vakalis, D. Hughes Hallett, D. Quinney & C. Kourouniotis (Eds.), *Proceedings of the Second International Conference on the Teaching of Mathematics (at the undergraduate level). July 2002*. CD-ROM Heronissos, Crete.

Malabar, I., & Poutney, D. (2002). Using technology to integrate constructivism and visualisation in mathematics education**.** On: I. Vakalis, D. Hughes Hallett, D. Quinney and C. Kourouniotis (Eds.) *Proceedings of the Second International Conference on the Teaching of Mathematics (at the undergraduate level). July 2002*. CD-ROM Heronissos, Crete.

Pierce R., & Stacey, K. (2002). Monitoring effective use of Computer Algebra Systems in K. Irwin & M. Thomas (Eds.) *Mathematics Education in the South Pacific.* (Proceedings of the 25th Annual Conference of the Mathematics Education Research Groups of Australasia, pp. 575–582.). Sydney: MERGA.