

First Graders' Use of Structure in Visual Memory and Unitising Area Tasks

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A cross-sectional descriptive study of 103 Grade 1 students from ten Sydney schools investigated the use of mathematical and spatial structure across 30 numeracy tasks. This report describes students' levels of structural development across two key tasks on visual memory and area as emergent, partial or identifiable structure. Lower-achieving students who lacked structure in their responses did not appear to be located on the same developmental path as other students. Qualitative analysis supported the findings of Gray, Pitta and Tall (2000) and Thomas, Mulligan and Goldin (2002) – that in the abstraction of mathematical concepts these students may concentrate on idiosyncratic non-mathematical aspects of their experience.

Widespread and early exposure to information technology has influenced the way children acquire mathematical concepts, highlighting the need for children to interpret such mathematical representations as models, pictures, diagrams, tables, charts and graphs (Diezmann & Yelland, 2000; Diezmann & English, 2001). Patterning and pre-algebra skills, interpretation and representation of data, and use of technology-based representations now form key components of primary mathematics curricula (Groves & Stacey, 1998; Board of Studies NSW, 2002). Acquiring these basic numeracy skills is proving increasingly difficult for those lower-achieving students who do not develop underlying mathematical or spatial structures.

Background to the Study

The development of mathematical structure has been described in terms of students' spatial skills and the importance of visualisation (Booth & Thomas 2000). Battista (1999a) refers to spatial structuring as:

... the mental operation of constructing an organization or form for an object or set of objects. It determines the object's nature, shape, or composition by identifying its spatial components, relating and combining these components, and establishing interrelationships between components and the new object. (p. 171).

Students' development of spatial structuring has been highlighted in studies of two and three- dimensional situations such as arrays of squares in rectangles, and cubes in rectangular boxes (Battista, 1999b; Battista, Clements, Arnoff, Battista, & Borrow, 1998). Related studies have documented the development of structure in the measurement of rectangles, squares and other two-dimensional objects (Outhred & Mitchelmore, 2000; Reynolds & Wheatley, 1996). Other lines of research show that, between Grades 2 and 4, most students learn to construct the row-by-column structure of rectangular arrays and also acquire the equal-groups structure required for counting rows and layers in multiples.

A common focus of modern mathematical learning theory is the structure of students' thinking and how well this reflects the structure of the concepts and relationships to be learnt (Hiebert & Carpenter, 1992; Lesh & Doerr, 2001). For example, children need to recognise mathematical structure in order to understand how the number system is organised by grouping in tens, and how equal groups form the basis of multiplication and

division concepts (Boulton-Lewis, 1998; English, 1999; McClain & Bowers, 2000; Mulligan & Mitchelmore, 1997).

Longitudinal studies of children's number concepts, including multiplicative reasoning, have highlighted the importance of mathematical structure (Mulligan & Mitchelmore 1997; Mulligan, Mitchelmore, Outhred, & Russell 1997; Mulligan 2002). Low achievers were more likely to produce poorly organised, pictorial and ikonic representations lacking in structure. These children lacked flexibility in their thinking; they were barely able to replicate models of groups, arrays or patterns that had been produced by others. Poor performance was attributed to students' primitive ideas that unitary counting can be used to solve everything and to their inability to visualise mathematical situations. A follow-up study of 24 of these students tracked to Grade 5, representing extremes in mathematical ability, indicated that low achievers lacked mathematical structure, and development was limited to pictorial and ikonic representations (Mulligan, 2002). Absence of any underlying structures persisted through to Grade 5. High achievers, however, used abstract notational representations with well-developed structures from the outset in Grade 2.

There is some evidence that some students do not develop structured images of critical mathematical concepts by Grade 2 and that, under normal classroom conditions, they may never develop them. If this is the case, staged models of cognitive development may need to be reconceptualised for 'at risk' learners. Previous research has looked for common developmental indicators of numeracy growth; comparing 'stages', 'levels', 'growth points' to determine better ways of assisting students' progress. A review of research on early mathematical development considers much more than counting and arithmetical knowledge that may occur in neat developmental steps (Gray, Pitta & Tall, 2000; Pirie & Keiren, 1992; Pirie & Martin, 2000; Sfard, 1991; Thomas, Mulligan & Goldin, 2002; Wright & Gould, 2002). However, a significant number of students do not progress on the same developmental path as other students. In the abstraction of mathematical concepts these students may concentrate on different objects or idiosyncratic non-mathematical aspects of their experience (Gray et al. 2000; Thomas et al., 2002).

Purpose of the Study

Previous studies have been based on small samples of 'at risk' students obtained incidentally in larger studies and have been inadequate to document their early mathematical experience, delays or impediments to typical mathematical development. It is not known how young children develop and apply pattern and structure across different contexts, or whether pattern and structure are essentially mathematical or related to spatial organization. This paper reports a cross-sectional study of Grade 1 students' use of pattern and structure in early numeracy. The study further identifies and explains the way students impose structure on mathematical situations through analyses of cases representing extremes in mathematical ability. In particular, this report explains how 'at risk' learners fail to show mathematical structure in two early numeracy tasks on visual memory and area.

Method

The interview sample comprised 103 Grade 1 students, 55 girls and 48 boys, ranging from 5.5 to 6.7 years of age at the time of interview. Subjects were drawn from nine NSW Department of Education and Training primary schools representative of six districts of metropolitan Sydney. The sample was representative of students from diverse cultural,

linguistic and socio-economic backgrounds. A sub-sample of twelve students representing extremes in mathematical ability (high ability and low ability) was selected for in-depth case study on the basis of the initial interview data.

Interview Tasks

Thirty tasks were developed on the basis of key mathematical concepts and processes categorised into Number, Space and Measurement strands for ease of referencing to syllabus outcomes and frameworks such as the Learning Framework in Number (NSW Department of Education and Training, 2002). The unifying feature of these concepts/processes was the interrelationships of their *mathematical and spatial structures*. In addition to assessing students' basic numeracy, all tasks required students to use or represent numerical structure such as equal groups, spatial organization such as rows or columns, or interpretation of a pattern. There were fifteen Number tasks including subitizing, counting in multiples, partitioning, multiplication (a combinatorial problem) and division (a quotitive problem). The six Space tasks included a task on visual memory, visualising and filling a box, and completing a picture graph. Measurement tasks investigated length, area, volume, mass and time concepts. These tasks integrated fraction concepts, conservation of length and students' own drawings of a ruler.

Procedures

Interview tasks and procedures were subject to pilot work; coding definitions and pilot videotapes were used for consistency and a 94% inter-rater reliability rate was found. Students were asked to explain and represent solutions by modelling, drawing and symbolising their mental images. Students were given the opportunity to provide alternate solutions and reproduce or modify their drawings. Interviews were segmented so that young students could complete tasks requiring drawing without time constraints. Follow up case study interviews required several segments of interview. All interviews were audiotaped, and case study interviews videotaped. The interviewer recorded students' response strategies and where necessary drew diagrams of models, noted explanations, gestures and finger movements. Operational definitions of strategy type were formulated from the range of responses elicited in pilot interviews and in accordance with coding employed in related studies (Mulligan et al., 1997; Mulligan, 2002; Thomas et al., 2002).

Analyses of Data

Quantitative and qualitative analysis describes task difficulty and individual patterns of response by task category. Students' strategies were coded for:

- Identification of mathematical features of the task such as structural /non-structural features of counting in multiples, grouping or using equal units (unitising);
- Interpretation and use/non-use of spatial structures such as using rows and columns in an array to show multiplication);
- Use of mathematical features in students' own drawings and representations (e.g., draws a grid to organise a solution);
- Use of idiosyncratic features such as drawings or explanations that do not assist in a correct solution process.

This paper focuses on a descriptive account of students' representations including diagrams, drawings, and explanations for two key tasks – visual memory and area. These tasks exemplify students' use, or lack of, mathematical and spatial structure.

Discussion of Results

The visual memory task required students to draw a triangular pattern of six dots. The analysis focused on whether students

- used features related to spatial organization such as three rows of dots evenly spaced to form a triangle;
- relied on unitary counting;
- looked for numerical and spatial patterns to assist in developing more efficient mathematical strategies.

For example, counting a pattern 1, 2, 3 to represent rows or the organization of three dots on each side of the triangle. The completion of the area task involves drawing squares of equal size as single units or by recognising the need to continue horizontal and vertical grid lines. In particular, this task shows whether the child can identify the organisational structure, the size of the unit (square) and the number of units required.

Table 1

Percentage of Responses by Level of Structure (N = 103)

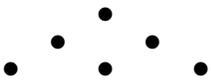
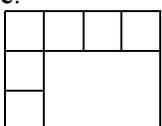
Task	Emergent structure	Partial structure	Structure
Visual memory			
Flash card with triangular pattern. Draw exactly what you see.	19	47	34
			
Area			
Someone has started to draw in some squares to cover this shape. Finish drawing the squares here.	16	40	44
			

Table 1 indicates that the visual memory task proved very difficult for students with only 20% of students giving correct responses. Interestingly, students could often remember the correct number of dots in the pattern but often lacked attention to the structure of the triangular shape. Of the 20% students giving an accurate response only 34% showed some structure, with 47% showing partial structure.

Figure 1 shows typical students' attempts to reproduce a triangular pattern of six dots from memory using some spatial structure (Figures 1.1 and 1.2) progressing to an accurate numerical and triangular pattern with reasonably good spatial organisation (Figures 1.3 and 1.4).

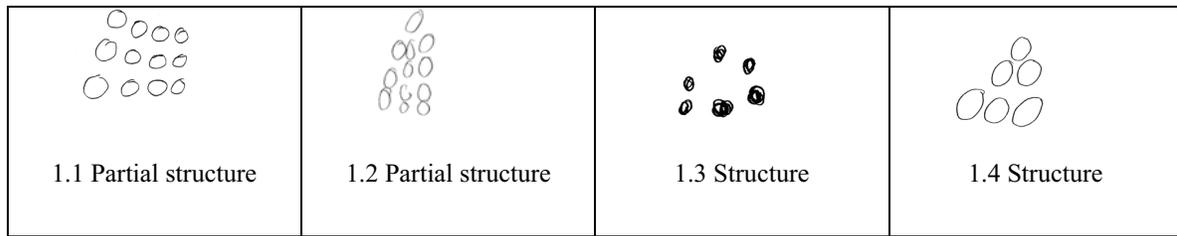


Figure 1. Typical examples of partial structure and structure for visual memory task.

Figure 2 shows drawings that are quite atypical: there is little awareness of spatial organisation, the structure of the pattern, or the correct number of dots although Figures 2.2 and 2.4 may be taken to indicate some attempt to represent the triangle shape.

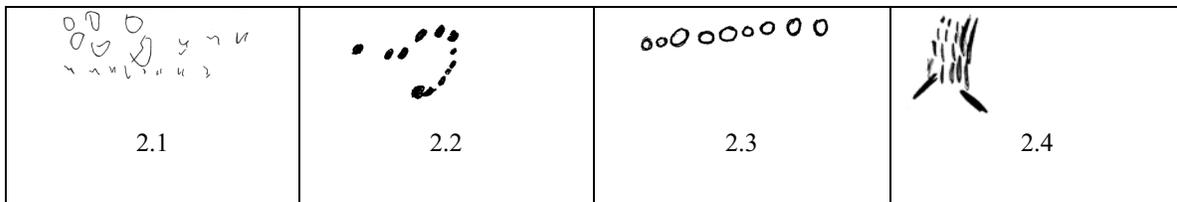


Figure 2. Responses indicating emergent structure for visual memory task for 'at risk' students.

Figure 2.1 shows an initial attempt to draw the correct number of circles, that is, six in a random arrangement. It shows recordings as squiggles that are unrelated to the triangular pattern of dots but there is at least a representation of the correct quantity of dots. When asked to explain the response the student mentioned six dots. However, there is no focus on spatial organisation, the pattern, or the triangular shape. Figure 2.2 depicts a curved sequence of dots that neither represents the shape nor the number. Interestingly, the student's explanation was that the curve resembled a triangle drawn as a rotation (90° left). In Figure 2.3, the student drew a row of dots bearing no relationship to the shape, pattern or quantity. Interestingly, several attempts made by the student to depict the triangular shape resulted in a variety of diagrams that simply showed circles. Figure 2.4 shows some triangular form drawn as a 'Christmas tree' as the student attempted to draw the pattern as vertical rows of five dots. There is little awareness of the structure of the pattern or the number of items in the pattern, although there is some indication of spatial organisation with equal-spaced marks.

For the Area task, 44% of students showed structure and 40% showed some partial structure. Some students showed emergent structure with signs of identifying unit squares albeit drawn in a disorganised manner. Figure 3 shows typical students' attempts to complete the task: a rectangular grid using squares. The examples show an increasing awareness of structure consistent with the findings of Outhred & Mitchelmore (2000). Figures 3.1 and 3.2 show partial structure as a correct number and size of squares in a border pattern, and correct number and alignment of individual squares respectively. These examples do not, however, show any indication that rows and columns are coordinated; although the equal groups structure of multiplication may be emerging. Figure 3.3 is a typical response showing awareness of structure and coordination of rows and columns (44% of correct responses).

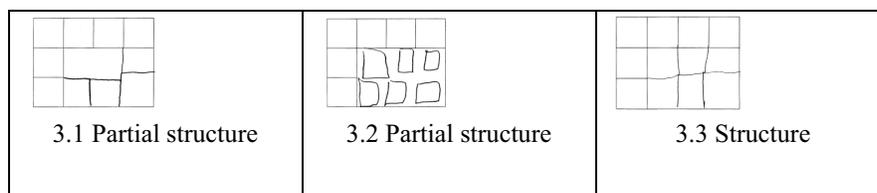


Figure 3. Typical developmental pattern for 'drawing units' task.

In contrast, Figure 4 shows some atypical responses to the same task, produced by 'at risk' students. The drawings lack numerical and spatial structure although Figure 4.2 suggests some awareness that the empty space needs to be filled.

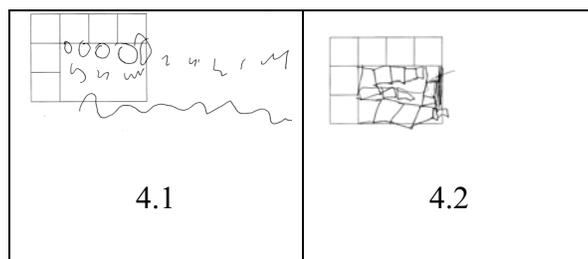


Figure 4. Atypical responses for 'drawing units' task for 'at risk' students.

Follow-up interviews and analyses of videotaped data of 'at risk' students indicated that the difficulty was not necessarily the comprehension of tasks, visual memory, the ability to draw or to count, but *the students' perception of structure*. Our analysis indicated that these students did not choose to completely ignore the structure but *their interpretation of the structure and the objects or shapes within it was in disarray*. There was also a wide variation between the 'at risk' students' responses, suggesting that *their images had been formed by influences unrelated to mathematical or spatial structure* (e.g., Figure 4.1 indicates some connection between a triangle and a Christmas tree). This supports the findings of Gray et al. (2000) that "in the abstraction of numerical concepts from numerical processes qualitatively different outcomes may arise because children concentrate on different objects or different aspects of the objects, which are components of numerical processing" (p. 401).

Our data obtained from 'at risk' students indicated that when these students concentrate 'differently' on objects they may not notice mathematical features and/or aspects of spatial organization at all; they reinterpret the 'objects' idiosyncratically. For example, Figure 2.3 depicts a row of small circles because the student focused on the image of dots or circles without noticing any numerical or spatial structure. In this example, the 'components of numerical processing' (i.e., the dots) interfered with the students' ability to abstract the quantity 'six'.

Implications

This analysis raises the question of why 'at risk' learners interpret mathematical situations without attention to structure and in an idiosyncratic way; how do they experience mathematics learning in everyday classrooms? Why is it that they do not 'travel' the same mathematical developmental path as others? How can we assist students to progress in a way that supports appropriate mathematical development?

This study supported our previous findings that some students do not develop structured images of critical mathematical concepts early and that, under normal classroom conditions, they may never develop them. If this is the case, staged models of cognitive development may need to be reconceptualised for ‘at risk’ learners. Although numeracy frameworks reflect the order in which strategies are likely to be used by children, this study highlights a long-standing issue for instruction and curriculum that not all children’s early mathematical knowledge develops along a common developmental path (Wright & Gould, 2002a). Furthermore, early numeracy has been dominated by traditional teaching methods focused primarily on arithmetical skills in the belief that given sufficient experience most children will eventually develop basic mathematical concepts. Teaching students to attend to structure in early mathematical situations may require professional development to not only detect this problem, but also assist the students to focus on all aspects of developing mathematical and spatial structure. This may be as simple as enabling students to visualise and record a simple pattern accurately.

We have provided some critical evidence to support further research into the development of mathematical and spatial structure in early numeracy. However, these data have not permitted immediate generalisation; nor have they provided a coherent picture of how ‘at risk’ students’ paths of development differ quite dramatically from the expected continuum of mathematical concepts and skills.

Further longitudinal investigation (using multiple case studies), could track the mathematical development of ‘at risk’ students from school entry through to primary level. Another aim would be to identify classroom influences that tend to promote or impede the development of structure in students’ images.

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References

- Battista, M.C. (1999a). Spatial structuring in geometric reasoning. *Teaching Children Mathematics*, 6 (3), 171-177.
- Battista, M. T. (1999b). Fifth graders' enumeration of cubes in 3D arrays: Conceptual progress in an inquiry-based classroom. *Journal for Research in Mathematics Education*, 30, 417-449.
- Battista, M. T., Clements, D. H., Arnoff, J., Battista, K., & Borrow, C. (1998). Students’ spatial structuring of 2D arrays of squares. *Journal for Research in Mathematics Education*, 29, 503-532.
- Board of Studies NSW (2002). Mathematics K-6 syllabus 2002. Sydney: Board of Studies NSW.
- Booth, R. D. L., & Thomas, M. O. J. (2000). Visualization in mathematics learning: Arithmetic problem-solving and student difficulties. *Journal of Mathematical Behavior*, 18, 169-190.
- Boulton-Lewis, G. (1998). Children’s strategy use and interpretations of mathematical representations. *Journal of Mathematical Behavior*, 17, 219-239.
- Diezmann, C. M., & Yelland, N. J. (2000). Developing mathematical literacy in the early childhood years. In N. J. Yelland (Ed.), *Innovations in practice: Promoting meaningful learning for early childhood professionals* (pp. 47-58). Washington: National Association for the Education of Young Children.
- Diezmann, C., & English, L. (2001). Developing young children’s multidigit number sense. *Roeper Review*, 24(1), 11-13.
- English, L. D. (1999). Assessing for structural understanding in children’s combinatorial problem solving. *Focus on Learning Problems in Mathematics*, 21(4), 63-82.
- Gray, E., Pitta, D., & Tall, D. (2000). Objects, actions, and images: A perspective on early number development. *Journal of Mathematical Behavior*, 18, 401-413.

- Groves, S., & Stacey, K. (1998). Calculators in primary mathematics: Exploring number before teaching algorithms. In L. J. Morrow & M. J. Kenney (Eds.), *The teaching and learning of algorithms in school mathematics* (pp. 120-130). Reston, VA: National Council of Teachers of Mathematics.
- Hiebert, J. & Carpenter, T. (1992). Learning and teaching with understanding. In D.A. Grouws, (Ed.) *handbook of research n mathematics teaching and learning*. NY: : Macmillan.
- Lesh, R., & Doerr, H. (2001). Symbolizing, communicating, and mathematizing: Key components of models and modelling. In P. Cobb & E. Yackel (Eds.) *Symbolizing, communicating and mathematizing*. Mahwah NJ: Lawrence Erlbaum.
- McClain, J., & Bowers, J. (2000). Supporting pre-service teachers' understanding of place value and multidigit addition and subtraction. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 279-287). Hiroshima, Japan: Program Committee.
- Mulligan, J. T. (2002). The role of structure in children's development of multiplicative reasoning. In B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.), *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, pp. 497-503). Auckland, NZ: MERGA.
- Mulligan, J. T., & Mitchelmore, M. C. (1997). Young children's intuitive models of multiplication and division. *Journal for Research in Mathematics Education*, 28, 309-331.
- Mulligan, J. T., Mitchelmore, M. C., Outhred, L., & Russell, S. (1997). Second grader's representations and conceptual understanding of number. In F. Biddulp & K. Carr (Eds.) *People in Mathematics Education* (Proceedings of the 20th Annual Conference of the Mathematics Education Research Group of Australasia, pp. 361-369). Rotorua, NZ: MERGA.
- NSW Department of Education and Training (2002). Count me in too: A professional development package. Sydney: NSW Department of Education and Training Curriculum Directorate.
- Outhred, L., & Mitchelmore, M. C. (2000). Young children's intuitive understanding of rectangular area measurement. *Journal for Research in Mathematics Education*, 31, 144-168.
- Pirie, S. E., & Keiren, T. E. (1992). Watching Sandy's understanding grow. *Journal of Mathematical Behavior*, 11, 243-257.
- Pirie, S. E., & Martin, L. (2000). The role of collecting in the growth of mathematical understanding. *Journal for Research in Mathematics Education*, 12, 127-146.
- Reynolds, A., & Wheatley, G. H. (1996). Elementary students' construction and coordination of units in an area setting. *Journal for Research in Mathematics Education*, 27, 564-582.
- Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.
- Thomas, N., Mulligan, J. T., & Goldin, G. A. (2002). Children's representations and cognitive structural development of the counting sequence 1-100. *Journal of Mathematical Behavior*, 21, 117-133.
- Wright, R., & Gould, P. (2002). Mapping overlapping waves of strategies used with arithmetical problems. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 418-425). Norwich, UK: Program Committee