The Development of Multiplicative Thinking in Young Children

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This paper describes developmental changes as children move from additive to multiplicative thinking. Five broad phases through which multiplicative thinking develops were synthesised from the research. These were labelled as one-to-one counting, additive composition, many-to-one counting, multiplicative relations, and operating on the operators.

Multiplicative thinking cannot be generalised in any simple way from additive thinking. This would not be a concern if additive thinking was sufficient for children to progress mathematically but it is not. Nunes and Bryant (1996) reiterate this point by stating:

> We have argued that children's first steps in multiplicative reasoning follow directly from their experiences in additive reasoning, but we have also tried to show that children who rely entirely on the continuity between the two types of reasoning will begin to make serious blunders in multiplicative tasks. So they must at some stage come to grips with the differences between the two kinds of reasoning. (p. 195)

This paper reports on part of a larger study (Jacob, 2001) that investigated the development of multiplicative thinking in young children. A premise of this study was that unless teachers can actually recognise the difference between additive and multiplicative thinking they would be unlikely to be able to help children develop the latter. The aim of the larger study was to answer the following questions:

- What is the difference between multiplicative and additive thinking?
- How can teachers recognise this difference in children?
- What are the developmental changes as children move from additive to multiplicative thinking?
- What helps children make the shift from additive to multiplicative thinking?

Part of the study included a small empirical investigation that explored the potential of a set of tasks for distinguishing between additive thinkers and children who are starting to think multiplicatively in a way that could be replicated in the classroom and thus be helpful for teachers. The results to this part of the study, which addressed the first two questions above, were reported previously (Jacob & Willis, 2001). This paper addresses the two latter questions.

Overview of Key Studies

A prerequisite to finding out how children develop multiplicative thinking is to understand the essence of what makes a situation multiplicative in the first place. An analysis of the mathematics of multiplication was carried out and a synthesis of the work of Davydov (1992), Clark and Kamii (1996), Boulet (1998) and Lamon (1996), in particular, led to the conclusion that it was identification or construction of the multiplicand and the multiplier within a situation, and the simultaneous coordination of these factors, that signified a multiplicative response to a situation.

A review of the literature on children's conceptual development as they learnt about multiplication and division was then carried out. Recent research into how children learn about multiplication and division has focused on the strategies children use to calculate a solution to different problems types. By analysing and categorising similar calculation strategies across a range of multiplication and division problems, researchers have made inferences about the kinds of conceptual structures or intuitive models children bring to such problems.

A number of key studies were reviewed and are summarised briefly here in chronological order. Kouba (1989) interviewed one hundred and twenty eight children in Grades One, Two and Three while they solved multiplication and division problems. One of her purposes was to classify children's solution strategies to ascertain the extent to which they were linked to the semantic structure of the problems. Also in 1989, Anghileri reported the results of her observations of the behaviours and successful solution strategies of 152 children aged from 6 years 8 months to 11 years 10 months as they carried out multiplication tasks. She described a progression of children's strategies from unitary counting or counting by ones, skip counting and repeated addition through to the use of a multiplication fact. In 1992, Steffe reported the results of a longitudinal teaching experiment with six children which commenced when they were eight years old. Steffe hypothesised that the number sequences constructed by the children would influence the way they thought about multiplication and division situations. He focussed upon children who had, or were constructing, the initial number sequence, the tacitly nested number sequence and the explicitly nested number sequence. He provides detailed reports of the thinking of one child in each group, Zachary, Maya and Joanna, respectively. Becker (1993) studied four- and five-year-old children in preschool to determine how they counted items that they had distributed in a many-to-one fashion, in order to find out the total number. More recent Australian research with children in the early primary years by Mulligan with Mitchelmore (1997) and with Watson (1998) followed. The 1998 Mulligan and Watson paper re-analysed the data from the 1997 study together with fresh data from a teaching project with Year Three children. They organised children's responses to multiplication and division problems in order to describe children's development of multiplication and division concepts. Finally, related work by Battista (1999) on children's use of rectangular arrays was examined.

Synthesis of Studies

The synthesis of the above literature suggested at least five broad phases through which multiplicative thinking develops. We have labelled these as *one-to-one counting, additive composition, many-to-one counting, multiplicative relations*, and *operating on the operator*.

One-To-One Counting

Children who count in a one-to-one manner are in this phase. They understand technically what they have to do to answer the *how many* question. Willis, Devlin, Jacob, Treacy, Tomazos and Powell (in press) state that "They will match the numbers in order as they point to or look at each object once, and they know the last number said answers the ‘how many’ question”. However children in this phase do not see the count as a permanent indicator of the quantity of the collection. According to Willis et al they do not trust the count and so do not count on to add two quantities and may think they will get a different count if the collection is rearranged or if they start in a different place. They also state that “Children who only learn to ‘skip count’ by reciting every second or every third number … may not realise that skip counting also tells you ‘how many’” (Willis et al, in press).

In multiplicative terms, the consequence of this thinking is that grouping makes no sense to them. They may know what it means to hand out three each, and so could be said to understand many-to-one correspondence, but they do not see the relevance of a many-to-one count.

This kind of thinking was well illustrated in Becker’s work. He presented two tasks that he called the *Hidden Items Task* and the *Needed Items Task*. In the *Hidden Items Task* the children were presented with four dolls arranged in a line. First they were given eight items and asked to give two items to each doll. The interviewer distracted the children from counting all the items by talking to them about the dolls. The items were then covered and the children were asked how many items were under the cover. The dolls remained visible. The task was then repeated with the children being given twelve items and asked to give the dolls three items each. In the *Needed Items Task*, children were presented with six dolls arranged in a row and were asked to give two items to each doll. However, this time the children were only given enough items for two dolls. The children were then asked how many items were needed to give two to each of the other four dolls. They were given the number of items that they said were needed, and were asked to distribute the items to the dolls. The interviewer asked the children whether the situation was fair or whether they needed more items and, if so, how many more. This was continued until they indicated that they did not need any more items. This task was then repeated for three items to each doll. Becker categorised the strategies children used to find the total number of items hidden or needed as follows: The responses relevant to this phase include:

- *One extra.* “Children said five items were needed, possibly because they knew they needed more items than four, the number of dolls.” (p460)
- *One-to-one counting*. Children pointed to each doll and counted by ones.
- *Separate groups counting*. Children repeated the number of items for each doll but did not integrate the count across the groups of dolls. They either said: “one, two; one, two; one, two; one, two” while pointing to places in front of the dolls or they said: “two, two, two, two”.
- *Rudimentary many-to-one count.* Children counted in a manner that was an attempt at a many-to-one count. “An example of this for a 3:1 trial was children counting 1, 2, 3 for the first doll, 4, 5 for the second doll, skipping the third doll, and 6, 7 for the fourth doll.” (p. 460)

Whilst children in this phase may be able to make the groups to represent a given multiplicative situation, they count by ones from the beginning in order to find out *how many*. As stated above, they may be able to say the number words in the *skip count* but they do not realise that skip counting must give the same quantity as counting by ones.

This idea was described by Steffe (1992). Zachary counted four rows of three blocks by threes because he knew the number sequence. However, when another row was added he could not count on the next three blocks. He had to start again and count by ones.

Battista (1999) found that many young children do not appear to see or use the row by column structure in a rectangular array to work out the number of squares. For example, he described how Katy, a second grader, was shown that a plastic inch square was the same size as one of the indicated squares in a 7 inch by 3 inch rectangle and asked to predict how many plastic squares would be needed to cover the rectangle completely. She placed her tiles in a spiral fashion first around the outside of the rectangle and then into the centre in order to work out how many squares would fit. She was then asked to predict the number of squares in a rectangle with partial grid lines drawn in. She counted in a spiral fashion around and into the rectangle again. She did not use the groups in the rows or columns to organise her count.

Children in this phase need to learn that a collection can be counted in different ways and the quantity stays the same. They also need to know that you can rearrange a collection and the quantity stays the same. These children need activities which challenge them to organise collections in a way that can be more efficiently skip counted rather than counting by ones. Children need to learn to recognise and use equal groups in rows and columns of arrays to say how many.

Additive Composition

When children understand that the count is a permanent indicator of quantity they are in this phase. They know that a collection can be rearranged or counted in a different ways and the quantity will not change. They know that given two collections, one with four and one with nine, they do not have to go back and count them all; they can simply count on from the four or the nine. Using shortcuts makes sense to them.

In multiplicative terms, the consequence of this thinking is that if children recognise equal groups they are in a position to take advantage of the groups to count more efficiently using skip counting and repeated addition. However, they may still need to lay the items in the groups out before they skip count or repeatedly add to find out *how many*. They do not yet understand that groups themselves can be counted. Their focus is on the multiplicand and they do not understand the role of the multiplier.

This idea was illustrated by Steffe (1992, p. 269) when he asked Zachary to work out how many times he would count if he counted a pile of twelve blocks by three. Even though Zachary skip counted to twelve he could not keep track of the groups of three as he went. In another task, Zachary made four rows of three. Without Zachary seeing, Steffe covered these along with three more rows. He then asked how many rows were added if there were seven rows hidden altogether. Zachary tried to work out how many blocks were hidden. He did not understand what it meant to find out how many rows were hidden.

Anghileri (1989) described several attempts by a child, JF, who was nearly nine years old, to work out the number of coins in a 6 x 3 array. The child was shown the array and then it was hidden from view. In the first attempt, JF repeatedly used the same three fingers on her left hand as she said, “one, two, three … four, five, six ... seven, eight, nine”. She used fingers on her right hand to keep track of the number of threes she had counted.

However, when she found she had three fingers on each hand she lost track of what the fingers on each hand were for. She then started again and counted rhythmically three fingers at a time proceeding from the left hand and continuing on to her right hand. She kept going in this manner until she was stopped at 27 by the interviewer and asked to think about the array again. She then made a third attempt.

> She now started again with three fingers of her left hand, 'One, two, three.' She clasped these together saying, 'One lot.' Now she extended the remaining two fingers of her left hand and one from her right hand saying, 'One, two, three … Two lots' She proceeded in this manner working across both hands counting in ones all the fingers she extended. 'One, two, three...six lots.' Now she went back to the beginning and successfully counted in ones all the fingers she had extended for grouping. (p. 373).

There seems to be some significance in children using different fingers across their two hands to represent the groups and not the same three fingers. It may indicate that they have not recognised or seen the equality of the sets and hence they have to separately represent each group as if they might be different to each other. A significant shift in thinking appears to be required to enable children to hold up one group of fingers and use it repetitively to represent the equal groups while using another set of fingers to keep track of the number of times that group has been counted.

Children in this phase need activities to help them recognise the number in each group, the number of groups and the total in multiplicative situations. They need to be able to describe multiplicative situations, for example in arrays, in terms of the number of groups and the number in each group without necessarily finding the total. They also need to learn to count groups simultaneously with the number in each group in order to find out *how many* in multiplication situations. They need to be able to count groups when given the total amount and the size of the groups in division situations.

If this group of children have concrete materials available to them when they solve simple multiplication and division problems, they can lay out the items as described in the problem and count, albeit they may count by two or threes. They do not need to keep track of the number of groups because the groups are out there already. They have only to focus on the multiplicand and count. They do not need to construct the multiplier for themselves. By not making materials available to children, as Steffe and Anghileri didn't, the task becomes a completely different one for them. Somehow they have to keep track of the number of groups. This may force children to construct the multiplier for themselves and learn to count groups and then to learn to count the groups and the number in each group.

Many-To-One Counters

When children understand that groups can be counted and that they can keep track of two things at once — the number of groups and the total of the number in each group — they are in this phase. They can hold two numbers in their head at once and double count. Children now know that they can represent one group and count repetitions of that same group.

Children double count in different ways: using fingers or using numbers. Steffe (1992, p274) described how Zachary used a combination of both when he counted by tens to find out how many piles of ten blocks could be made from 87 blocks. He carried out two parallel counts but only needed to represent the multiplier with his fingers in order to keep track of where he was up to. With more simple situations or problems children may carry out two parallel counts that use the numbers alone. They count the repetitions of the multiplicand and, as they go, they count the number of groups, that is, they keep track of the multiplier as a way of knowing when to stop counting. For example, "5 for 1, 10 for 2, 15 for 3, 20 for 4" (Mulligan & Watson, 1998, p. 74).

Children may initially think of the multiplier as the number which tells them to when to stop counting. In order to move on, the role of the multiplier needs to change for children. The multiplier needs to come to the forefront so that the multiplicand and the multiplier are coordinated prior to the count in order to produce a multiplication operation. Steffe (1992), in illustrating what it means to multiply, talks of the need to coordinate the units (factors) prior to carrying out the calculation. He describes how Maya did this the very first time he worked with her (1992, p. 279). Steffe had a red piece of construction paper, several congruent rectangular blue pieces cut so that six would fit on the red piece, and equal sized orange squares cut so that two would fit onto each blue piece. He gave Maya the red piece and three blue rectangles. He asked her to work out how many blue pieces would fit on the red piece. After she said six he removed the three blue pieces she had placed on the red piece, and placed two orange squares on one blue piece. He then asked her to work out how many orange squares would fit on the red piece without using the actual paper pieces. Maya looked straight ahead, mouthed number words, then said twelve. She explained how she worked it out by tapping the table twice with each of six fingers while saying the number words: "1, 2 ... 3, 4 ... 5, 6 ... 7, 8 ... 9, 10 ... 11, 12". Steffe inferred from her actions that she filled each symbol for a blue piece with a symbol for two orange pieces because she could take the two orange pieces as a unit. However, according to Steffe, although Maya could coordinate units for multiplication she could not coordinate units for division. She needed to double count to work out a division. Children need to learn to coordinate the multiplicand and the multiplier for division as well to be in the next phase.

Children in this phase generally carry out the double count for multiplication and division completely separately from one another. They do not fully understand the relationship between the number in each group and the number of groups and the total in multiplicative situations and so are not in a position to move flexibly between multiplication and division. They cannot consistently use the inverse relationship between multiplication and division or the commutative property of multiplication.

Children need to learn to identify the number in each group, the number of groups and the total in a range of multiplicative situations and come to know that it is the unknown quantity that makes the situation a multiplication and/or a division. This enables them to use the inverse relationship and move flexibly between multiplication and division.

Children also need to understand part-part whole reasoning with groups in both multiplication and division situations.

An important finding from this research is that one cannot assume that children who can coordinate units, or think of multiplication situations in a binary way, are going think about division situations in the same way.

Multiplicative Relations

When children come to know that multiplicative situations involve three aspects: groups of equal size (a multiplicand), numbers of groups (the multiplier), and a total amount (the product), and can coordinate the grouping structure in both multiplication and division problems prior to carrying out the count, they are in this phase. Children know which number tells them how many in a group and which number tells them how to operate on that group. For example, they know whether to find six times the amount in the group, or to find one sixth of the amount. In multiplicative problems one of those aspects is missing and requires an operation to work it out. Having a deep understanding of the roles of these numbers and the relationship between them also enables them to understand and use the inverse relationship and commutativity. Part-part whole reasoning with groups also enables children to use the distributive property of multiplication over addition.

Joanna, in Steffe's 1992 study, demonstrated that not only could she coordinate units prior to calculating in both a multiplication and a division situation but that she understood and could use the commutative property of multiplication when the materials were there to see. Steffe carried out the activity with the coloured paper pieces previously described for Maya. When asked how many orange pieces fitted on the red square, Joanna quickly said that it was twelve. In explanation she said, "Well, six plus six is twelve, and *each two blocks fit on one big block*, and that makes twelve." (1992, p. 292) Steffe inferred that "Joanna took six units of two as a given for further operating, splitting and doubling whereas Maya took the six units of one as a given and then substituted a unit of two for each unit of one" (p. 292). It seems whilst Maya counted two for each blue piece, Joanna knew that there were six twos, but that she could work it out by saying, "I've got six once; I've got six twice. So six and six are twelve." She could see in that situation that two, taken six times, was the same as six, taken two times. To find out how she coordinated units in a division situation, Steffe asked Joanna to find how many of twelve marbles would go into each of three cups, with an equal number in each cup. She answered, "four," and said, 'because four in that one and four in that one would be eight, and four in that one would be twelve." (p. 292) Steffe asked her if she tried anything else and, laughing, she said, "I tried three but it wouldn't work" (p. 292). Steffe describes her unit segmenting scheme as anticipatory with the units being available to her prior to operating. In this case Joanna must have interpreted this situation as

3 somethings makes 12' or '3 x ? = 12.

Mulligan and Watson also describe typical responses to problems indicating that children are starting to use commutativity, inverse relations, and part whole understanding. These responses include: "6 times 4 is 24 so 12 times 4 is 48. 72 divided by 8 … 9 eights are 72

3 by 7 is 21 so 42 divided by 7 must be 6." (1998, p. 77) Although children think multiplicatively in this phase there is still more to be learned in order that they be fully operational thinkers.

Operate on the Operator

This phase of thinking was not the focus of this study and hence the research about operational thinking was not described. It is briefly mentioned here to signify that there is still more to know about multiplicative thinking than that indicated in the previous phase. When children can operate on variables in algebraic situations and operate on operations they are in this phase. It would mean that they could multiply the multiplier in the way described by Schmidt and Weiser (1995) in the multiplication problem type they call the *structure of composition of operators*. They give an example of this type of problem: *During the first year of life Otto the elephant trebles his weight at birth. In his second year he doubles his weight. What multiple of his weight has he got at the end of his second year of life?* (p. 60) Also among other things it would mean that they could flexibly use factors in order produce a multiplication number sentence that was easier to calculate.

Conclusion

In order to answer the research question *What are the developmental changes as children move from additive to multiplicative thinking?* an analysis and synthesis of the existing research literature was undertaken. Five broad phases through which multiplicative thinking develops were identified. These were labelled as *one-to-one counting, additive composition, many-to-one counting, multiplicative relations* and *operating on the operators*. For the purposes of this study, the *additive composition* and the *many to one* phases were considered to involve additive thinking. The *multiplicative relations* phase indicated when a child was first thinking multiplicatively and the *operating on the operators* was the phase where children were considered to be fully multiplicative thinkers. This synthesis also suggests that there is a transitional phase between additive and multiplicative thinkers and that is the *many to one counting* phase. At the conclusion of each phase descriptive information was included to help answer the question "What helps children make the shift from additive to multiplicative thinking?"

The phases of thinking as they develop multiplicative thinking will influence the types of responses children are capable of making as they solve multiplicative tasks. From children's responses, useful inferences can be made about what they will and will not understand and be able to do. Willis et al (in press) suggest that recognising common patterns of thinking should help teachers to interpret children's responses to activities and to understand why children seem to be able to do some things and not others, and importantly what to do about it and when. A major aim of this study is for teachers to be able to recognise multiplicative thinking when it occurs, to recognise progress towards it and to design opportunities that enable children to progress. This would also involve teachers coming to know what helps children make the shift from additive to multiplicative thinking. Further developmental work may be necessary before this information can be made fully accessible to teachers.

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