Percentages: A Foundation for Supporting Students' Understanding of Decimals

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This study involved a six-month teaching experiment aimed to improve children's understanding of decimal fractions. Data indicated that the learning activities based on the work of Moss and Case (1999), which promoted the use of percentages as a visible introductory representation, significantly influenced the students' emerging understanding of decimal fractions as quantities. Sense making was maintained as the students increasingly translated across rational number representations—using and extending their informal understandings into more formal decimal knowledge.

Difficulties with decimal fractions and the tenacious decimal misconceptions students construct are well documented by New Zealand and overseas researchers and educators (e.g., Boufi & Skaftourou, 2002; Irwin, 1999; Padberg, 2002). The difficulties identified include a lack of understanding of the decimal place value system and an inability to quantify decimal numbers—leading to problems applying relational knowledge to decimals, difficulties with benchmarking fraction equivalents, and a lack of number sense when using decimal numbers in operations (Storey, 2001). An added difficulty resides in the timing of the introduction of decimals. This typically occurs in the middle school years—at a point when according to Beswick (2002), many students and teachers no longer expect mathematical activity to directly link experientially to problems in the real world. In such situations many students develop procedural competence with decimals not as a sense-making activity, but rather within a rule-based and algorithmic framework.

While many research studies on decimals have centred on the role of specific instructional activities (Hiebert & Wearne, 1986; Stacey, Helme, Archer, & Condon, 2002) and the nature and types of student misconceptions (Stacey & Steinle, 1998), increasingly researchers are advocating the need for a teaching approach which takes cognisance of the informal knowledge of the students and the strategies they use (Boufi & Skaftourou, 2002; Irwin, 2001). Sense making is promoted through active engagement in communication in the classroom and gradual mathematisation of the student contribution. In this study a similar approach was used to intentionally focus students aged nine and ten on the connections between known real world quantitative representations, the language used to describe them and the written symbols. Classroom activities were designed to refine and extend students' informal rational number concepts.

The teaching experiment based initially on the work of Moss and Case (1999) promoted the use of percentages as an introductory external representation for students' emerging understanding of decimals. In accord with Goldin and Shteingold's (2001, p. 8) contention that "the power and flexibility of the internal representations, including the richness of the relationships among different kinds of representation", is central to conceptual understanding, it was envisaged that percentage tasks would provide a rich connective base for other representations of rational number. While in the first instance, particular representations offer students 'thinking spaces', ultimately, translating across a range of representations is needed for the construction of robust rational number concepts (Lamon, 2001). Thus, the focus of the teaching experiment shifted from student attainment of specific sections of rational number knowledge towards growth of a more global conception of the rational number system (Moss & Case, 1999).

The theoretical framework was derived from Cobb's (2000) emergent perspective. From this perspective the construction and reconstruction of decimal concepts result from both individual and collective mathematical activity in a relationship of reflexivity in which neither is given more significance than the other. In such learning environments the emergence of taken-as-shared meaning for rational number concepts is constrained or enabled through individual contribution as students participate in social activity in the classroom. The presentation of individual student's understandings are therefore embedded in the emergence of taken-as-shared meaning of decimal fraction understandings which evolved within the social context of the classroom.

Research Design

The study involved a 6-month classroom teaching experiment (Cobb, 2000) conducted at a large inner city primary school. Students represented a range of ethnic backgrounds and came from predominantly high socio-economic home environments.

A collaborative partnership between the researcher and the teacher supported the development of a hypothetical learning trajectory for decimal understanding. Following individual interviews, four students were selected as case studies to represent the range of decimal misconceptions common to students within the middle school age group. Data were collected from case study student interviews, 15 lesson observations, and classroom artifacts.

Analysis of data used a grounded approach identifying codes, categories, patterns, and themes. These were used in conjunction with participant dialogue in order to give voice to the students as they participated in classroom activity designed to support their construction of decimal concepts.

Although the learning trajectory was used to establish initial learning activities the recursive and non-linear paths taken by students in their construction and reconstruction of decimal concepts subsequently influenced the choice of further activity in the teaching and learning cycle. As a result of on-going data analysis from classroom observations and student interviews, the teacher and researcher revised and modified the instructional sequence.

Results and Discussion

The overarching purpose of the teaching experiment guided the selection of an initial set of activities. These activities were chosen to support students' construction of in-depth, rich conceptual knowledge of decimal fractions as quantities that could then be used as reasoning tools to support translation across other rational number representations within a range of problem solving situations. It was anticipated that this could be done through refining and extending existing informal understanding of proportional reasoning using a unidimensional representation.

Percentages and Proportional Reasoning

Percentages were selected as an introductory scaffold to other forms of rational number, on the basis that the students already had well-developed informal understandings of both proportions and numbers from 1 to 100. Moss and Case (1999) argue that by beginning with percentages the need for students to engage in the complexities inherent in comparing ratios with different denominators are delayed—and that every percentage has an easily seen decimal or fractional equivalent. Furthermore, the teacher felt confident that the students already had informal knowledge of percentages through their daily experiences. The teacher reported that the students were frequently observed to use percentage terminology in their everyday language.

The students were introduced initially to rational number through activities using water in clear cups of varying sizes. In collaborative groups the students explained the strategies they were using to estimate the "fullness" or percent of the height of water in the cups. They demonstrated no difficulties in assessing the levels in global proportional terms or in assessing proportional rather than absolute matches as evidenced in the following sample:

Sara: Full to capacity but that one is half full, fifty percent full.
Adam: Hundred percent full or it could be just a little bit less like 99%.

These initial activities were designed to maximize both the students' informal understanding of ratio and numerical splitting. During introductory activities as students worked with water assessing the quantity in containers they were observed to spontaneously use what Moss and Case (1999) term visual motor splits, halving, and composition. This is illustrated in the following example involving the students assessing quantity. They used hand movements to indicate partitioning of the quantity coupled with explanations involving numerical halving (e.g., 100, 50, 25), and composing and de-composing of numbers to 100.

Jane: If you have half you have 50% and again a quarter so that would be 25%.
Sara: So if you add them both together you make 75% so now our cup is three quarters full again.

In order to extend students' thinking contextual metric problems were introduced. These problems required precise calculations while simultaneously connecting representations of number, ratio, and measure. Again the students spontaneously used the previously observed numerical halving strategy but importantly were prompted to extend their strategy explanations to include the use of portions of 10%. Also prior knowledge of other rational number concepts was evident. When translating across percentages students were intuitively seen to benchmark to decimal and fraction equivalents. This is illustrated as a student draws a diagram of a 1.5 litre coke bottle and explains his thinking using his hand to illustrate visual motor splits.

Eric: Then like 50% of 1000mls is 500mls and then 250mls so that would be 750mls.
Jane: Why?
Eric: You know 50%, that is half of 500mls is 250mls so they each get 250mls out of the point five and you just take away the one to get the point five so they each get 750mls.
Brenda: Why?
Eric: Because half of the point five is 250mls and half of one litre is 500mls and then you add the 250mls and the 500mls and so you have 750mls.
Brenda: I am just not sure if point…

Eric: Point five is half.
Brenda: So you just take half of that and half of that?
Eric: It's half again.

The percentage problems embedded within the context of liquid served as a rapid and serviceable scaffold, enabling students to confidently solve ratio/measure problems. The students also subsequently used as concrete or pictorial representation, a proportion of liquid in a container or the enlarged number line on the floor to justify their reasoning. In addition, these were used as mental images in explanations providing experientially real taken-as-shared objects that enabled the students to "fold back" or "drop back to" and thus support subsequent thinking activity (McClain & Cobb, 1998).

Linking Proportional Representation on a Number Line to Decimals

At this point in the teaching experiment a large number line was drawn on the classroom floor (with numbers placed exactly one metre apart and calibrated in tenths and hundredths). The aim was to build on the students' confidence with percentages in order to make explicit the link to two-place decimals. Again problems were designed to capitalise on measurement contexts that would be familiar to the students. For example, running races of short distances and competing to jump the longest distance, measured arbitrarily by students are commonplace activities in the playground. In the classroom situation the student was instructed to run a short distance along the number line and then stop. The students understood that the two adjacent numbers represented whole numbers, the first marking the distance which had been run (e.g., 3 metres) and the second marking the end of the next complete metre they were running towards (4 metres). The students were required to calculate the distance covered in completed metres and then calculate both, what percentage of the next whole metre they had run and what percentage they needed to run to complete the metre (e.g., 23% run and 77% to run). The students spontaneously transferred strategies used to estimate water quantity to estimate distance and again flexibly applied their informal knowledge of fraction and decimal numbers to their percentage equivalent. They did this in order to justify explanations of a proportional amount of distance shown on the number line as illustrated in the following example:

Eric: Yeah 2 metres and 75, oh 2.75.
Fay: So it was 2 metres and 75%.

In order to ensure explicit linking of the equivalent percentage and decimal value the teacher frequently revoiced a student statement. This not only made the explanation accessible to other students but also provided opportunity for the teacher to reinforce both the continuous nature of decimals and the need for students to state the unit referent:

Eric: 14% or point one four.
Teacher: [She records in large writing .14 on the whiteboard.] So you say that he has 14% of the next metre still to walk, so what you are saying is that he still needs to walk point one four…yes point one four of the next whole metre…that is the same as 14% of the next whole metre.

In this way a norm was established in which both listening students and the teacher expected explanations which included a description of a percentage value, its decimal equivalent and the unit referent. In addition, a bridge to the recording of decimal symbols was also provided through the explicit recording of the student's explanation.

Quantitative understanding is critical if students are to apply number sense when using decimal symbols in operations. Moreover, if decimal symbols are to be used as mental referents, they need to be tied to experientially real objects as quantities and understood as one whole unit. In this study, these links were assisted by the teacher's explicit revoicing of explanations to construct links between the proportional amount as a percentage, its decimal equivalent and its recording as decimal notation as demonstrated:

Teacher: So I just heard Jane say 3 metres and 47% of the next meter and—then I heard Eric say that is 3.47.

At this point, the teacher records 3.47 establishing a pattern for subsequent recording of decimal notation.

However, in order to explicitly reinforce for the students the notion of one whole unit (in this instance a metre) as the referent unit, the teacher chose to use the word percentage as a scaffold. The teacher recorded and underlined 'per cent' when a student used it to describe a distance and asked the students to clarify its meaning. The teacher responding to a student's interjection of "out of", stated in response to another student's explanation of "per hundred", probed further by asking "out of what?" She then revoiced to extend the answer, "so 33% means you have 33 out of 100" and added with emphasis the statement "of one whole metre". Placing the emphasis on the 'one' focused the students' attention on the notion of one whole unit and reinforced the need to state the referent unit. Subsequently, the teacher explicitly maintained links between the language of the referent and the symbols that represented the referent, as the following extract illustrates.

Teacher: I saw you hesitate, so I saw you write 0.71 and then you looked like you wanted to write something else?
Eric: Yeah metres, I was going to write metres but I didn't cos it isn't a whole metre.
Teacher: Can you say 0.71 of a whole metre?
Eric: Oh yeah.

The affirmative nods from peers and the probing question effectively prompted rethinking and the student then emphatically recorded the word metre next to 0.71.

From this point on in the study the students also consistently asked their peers for specific clarification of referents used in explanations of actions on quantities.

Brenda: What do you mean by minus 100? Or do you mean 0.100 or 100 metres or what?

They also explicitly modeled the referent unit in their own verbal or recorded explanations. This is shown in the follow example involving a student's report of his group's strategies:

Eric: Yeah and then from 2 and 41/100 you minus the 41/100 and that would equal just plain 2.

At this point he then recorded the number 2 and underneath it in large writing added the word 'wholes'.

The Number Line as a Concrete Representation

In this study the number line provided a powerful visual concrete representation. The teacher used it to extend the students' informal knowledge of partitioning to include a perspective that not only conceptualized the continuous nature of decimals but also reconceptualised the unit, so that the students viewed its fractional parts in relation to the unit whole. Each metre section of the number line served as a concrete representation for the students of one hundred percent. It was also re-described by the students as a fraction of 10/10 and 100/100 and as a decimal referred to as 'one point'—reinforcing the concept of the unit whole. This is illustrated during an explanation given by a student in which she had become confused and a second student suggested the use of 'taken as shared' knowledge to denote the unit whole:

> Brenda: Maybe to make it look a bit easier, like to make it clearer you need to put a dot after the one so that we all know that you are talking about one point.

In addition, the use of the number line in contextual problems served as a means to confront common decimal misconceptions—namely the denseness between decimal numbers and the role of zero as a placeholder (Irwin, 1996). In order to do this, the teacher asked a student to walk a very short distance along the number line and then stop. The watching students were asked to quantify the distance covered and the distance to the next full metre. During the student explanations, teacher questioning lead to a "fold back" in thinking when a student described the distance walked incorrectly as 761.4 rather than 761.04.

> Teacher: So are you saying that he had walked 40% of the way?
> Eric: Oh no that's only 4 centimetres and…oh I can't explain it.
> Fay: Well now I think it might be 761.04. See the zero comes in because if you said 761.4 it's another way of saying 761.40 like 40% and so if you don't want to say 761.4 you have to put another zero in front of the 4 otherwise it will mean 40%.

However, misconceptions are known to be extremely robust. The teacher recognises this in the following episode when in response to continuing puzzlement of many of the students she reconstructs another context, "folding back" further by picking up a clear cup and probing:

> Teacher: If I walked 4% of the way Fay said that is the same as 40%. Now let's think of this cup if you fill it up 4% is that the same as 40%? If you want to fill it right up to the top how much do you have to put in if you have 4%?
> Sara: 96%.
> Teacher: If you have a cup that is filled 40% how much do you have to put in to fill it to the top?
> Fay: You have to fill it 60%.
> Teacher: So have you walked 40% of the way of 4%?
> Fay: Just 4%.
> Teacher: So can you say 0.04 is the same as 0.40?
> Fay: No because 0.40 is 40% but 0.04 is just 4%?

In this example the teacher had responded by not just "folding back" but also "dropping back"—picking up the cup had constituted a new beginning in the discourse (McClain & Cobb, 1998). The drop back in context and translation across representations had served a useful purpose; causing sufficient conflict to potentially support the restructuring of concepts of fractional decimal numbers (below point one).

Translation Between Modes of Representation

Connecting across and between subconstructs of rational number supports deeper conceptual understanding. As the teaching experiment progressed it was evident that the students' flexibility in and between modes of rational number representations increased as they made connections between the problem contexts, their informal and formal rational number concepts, and the notation system. The added dimension of flexible translation between representations provided a powerful thinking tool that the students frequently used to check their reasoning and the reasoning of others. This was evident as a group of students worked collaboratively on a problem which required exchanging NZ$1000 given that NZ$1 was equivalent to Australian $0.88 or American $0.47 and English £0.232.

Stefan: Geez look at that, what a rip off, you only get 232 pounds, what about America?
Eric: It's about a half, no a little bit less than a half, so less than $500.
Brenda: Yeah but with the English one you get about a quarter so I would go to Australia.
Stefan: So you get like three-quarters so like $750?
Eric: More because it's 0.88 so you only need 2% more and you have 90% so you actually get closer to $900. You can just estimate it in your head. It's closest to percents.
Brenda: And fractions.

Translation between representations was not only a means to check and clarify the measurement of proportions but also often provided sufficient challenge to the students' thinking to bring about reconstruction of decimal fraction concepts. In the following extract this is illustrated when a problem solution is recorded as 2 metres and 75% of the next metre or 2.75 metres and this is questioned by a listening student.

Sara: Wouldn't you write that as 75 over 200?

The student who recorded the solution repeats the problem context but renames the 2.75.

Jane: So you are saying she walked two and three quarter metres?

The re-describing of 2.75 as two and three quarters presents a conflicting representation to the first student who then rethinks her prior knowledge.

Sara: No wait I think maybe it is 75 out of 100 because if you use 75 out of 200 then you are not talking about three quarters of the way like then I think you are talking about much less.

In this way, a translation across representations caused the student to reconceptualise her developing understanding of rational number as quantity and provided her with a means with which she was able to maintain sense making.

Conclusion

The teaching experiment was designed to build on the students' informal rational number understandings and in particular to use their prior knowledge of percentages as a rich connective foundation for translating across decimal and fractional rational number representations. More specifically, it was developed to support middle school students' emerging understandings of decimal fractions as quantities and the written symbols used to represent them. The description of the learning activities presented in this paper, while only a small sample of those used in the teaching experiment, demonstrate that not only did the activities successfully extend the students' understanding of rational number beyond construction of decimal fraction understandings—they also intentionally supported understanding of rational number as an integrated, multi-leveled and multi-connected system.

The use of percentages as an introductory unidimensional representation embedded within contextual problems involving ratio/measure of water supported the students to construct a robust and effective overview of rational numbers. The students during discussion and activity spontaneously applied prior knowledge of proportional thinking in activity and this provided the teacher with a bridge to scaffold understanding of not only the continuous nature of decimals but also the notion of the unit whole and the referent unit.

Also, the establishment of classroom norms that supported students' understanding was clearly articulated in the shared expectations for active listening, questioning, clarifying meaning and making predictions and justifying conjectures. Through the teacher's intentional use of the student explanation, decimal symbols became rich mental referents for the students, tied to experientially real objects as quantities and able to be used in sense making activity. Within the classroom individual students appeared to construct and reconstruct rational number understandings through taken as shared meaning of decimal fraction understandings which had evolved in a learning environment that was neither wholly individual nor wholly social.

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