Investigations Into the Introduction of Logarithm Tables in Victoria

Chris Barling
Swinburne University of Technology
<cbarling@swin.edu.au>

One of the first major changes in mathematical technology used in upper secondary school mathematics in Victoria was the introduction of logarithmic tables in the early twentieth century. Information about this change and the reasons for it is difficult to come by, yet much can be gleaned from such examination papers and official publications as survive. Understanding of this process is of interest as an early example of the introduction of technology into the mathematics classroom, a phenomenon of particular interest in our own times.

This paper represents an early report on one phase of an on-going study of the technologies used in the teaching of mathematics in the upper secondary school in Victoria. Its driving force is the need to understand the processes that lead to each change and that inform its success or failure as an innovation: in this respect by understanding our history we are better placed to plan and manage the increasing pressure of relentlessly changing technologies in our own time.

At the time of writing the author is at an early stage of this particular investigation: this paper is a progress report on the detective work of discovering what took place and when in a significant moment in our technological history.

The Earliest Victorian Matriculation Examinations

University education began in Victoria in 1855, and matriculation (university entrance) examinations were conducted at least from the end of 1857. To begin with there were three mathematics subjects: Arithmetic, Algebra, and Euclid (or Geometry). In Arithmetic, there is a progressive mixture of routine calculation and practical problems:

> 2. Express 5s. 10d. as the fraction of £1; and reduce it to an equal fraction whose denominator shall be 168. (Arithmetic Examination, 1857)

Within a decade it had become routine to include long and tedious exact calculations in form like the following:

> 1. Write down in words the quotient and remainder obtained by dividing three billions four millions five hundred and six thousand and seven by fifty thousand millions seven hundred and nine. (Arithmetic Examination, 1865)

As well as other numerical calculations to be done to a specified accuracy:

> 5. Divide the product of 25.378 and .00246 by .0000927 to four places of decimals.
>
> 9. Extract the square root of 0.144 to four places of decimals. (Arithmetic Examination, 1865).

From 1870 the assumption of exactness was made explicit:

> Every result must be reduced to its simplest form. Where the answers are not exact, they must be worked to three places of decimals.

Gradually towards the end of the century there was a reduction of interest in the more tedious long exact calculations and an increasing attention to more practical problems that required approximate answers of a specified accuracy. But the intention throughout this period is clearly that students should demonstrate their mastery of the standard (pen-on-paper) algorithms for multiplication, long division and the extraction of square roots.

The only alternative techniques available at the time were those used commonly by engineers, navigators and some scientists who required rapid and frequent calculation; and in these professions the customary practice was to use logarithms. For rough calculations in the field, a slide rule was used, which would give an approximate answer with perhaps three or four significant figures accuracy; and for more accurate desk calculation, tables of logarithms were used (Barling, 2003a). In the nineteenth century these were always quite substantial volumes: the most common, Chambers' Tables, was a book of well over 400 pages (Pryde, 1878; Barling, 2003b): the table of logarithms alone occupied 200 pages, giving each logarithm to seven figures. Generally the use of these tables would give results accurate to six or seven significant figures. The use of logarithms was regarded as a necessary professional skill, but it was taught within the confines of professional training, usually through a sort of apprenticeship. There was a clear dividing line and mutual suspicion between the practical work of professionals such as engineers and navigators, and the exact academic work favoured by schools and universities.

Change in the Late Nineteenth Century

The educational situation was, however, far from static: there were powerful movements for reform throughout the last decades of the century, not least in mathematics (Blake, 1973; Sweetman, 1922). A reform group formed originally in Britain in the 1870s as the Association for the Improvement of Geometrical Teaching had by 1897 become the Mathematical Association, which promoted and influenced the teaching of all school mathematical subjects both in Britain and Australia (Griffiths & Howson, 1974). The Association's ideas were readily acknowledged in Victoria: from 1905 all of the Public Examination syllabuses contain the exhortation

> The papers in mathematics will be set in general accordance with the recommendations contained in *Teaching of Elementary Mathematics: report of the Committee appointed by the Mathematical Association*, published by G. Bell & Sons.

There had been some reform as far back as 1880, when all of the matriculation examination syllabuses had been rewritten; but the new forms are less than transparent as to intentions, giving only references to then-standard textbooks. Perusal of these volumes shows little change in the arithmetic procedures covered or the manner of their execution. In particular, there is no use of logarithms and little of other forms of approximate calculation in the Arithmetic text; and in the Algebra texts logarithms appear only in the later section excluded from the Victorian syllabus. Similarly in the next syllabus revision of 1898 there is still no inclusion of logarithms as a topic in school Arithmetic or Algebra.

Change first took place in Geometry, where from 1880 the Honours paper in Geometry included a section in Trigonometry:

> Honours: The substance of Euclid, Books I–VI. Trigonometry not more than is contained in Todhunter's Larger Trigonometry, chapters i – xvii, omitting chapters ix and xii.

"Todhunter's Larger Trigonometry" is Plane Trigonometry by Isaac Todhunter (1869). Within this volume logarithms are discussed in chapter x (after the omitted chapter ix, on the construction of logarithmic tables) and the use of logarithmic and trigonometrical tables in chapter xi. In pre-calculator days studying trigonometry without the use of tables would have been inconceivable, and it is indeed here that the use of logarithms makes its entry (by implication) into the Victorian school syllabuses. In the next syllabus revision of 1898 the topic is explicitly mentioned, again in the Trigonometry section of Geometry Honours:

> Trigonometry: The measurement of angles. Trigonometrical ratios of one, two or more angles. Use of logarithms. Relations between sides and angles of a triangle. Solution of triangles. Heights and distances. Properties of triangles. Inverse trigonometrical functions. Elimination.

But this then raises questions as to how the topic could be taught in schools, and even more how it could be assessed in public examinations. At this time the only commercially available books of logarithmic tables were serious and probably expensive volumes of several hundred pages (Barling, 2003b): one cannot imagine students being required to own a copy, or schools or examination authorities supplying class sets.

If we turn to the actual examination papers we see that in practice the use of tables and logarithms was largely ignored, at least in the formal examinations. Here, for example, are the five trigonometrical questions from the paper of November 1899:

> 7. Explain clearly why there are four values for the sine of half an angle when the sine of the angle is given. Show that sin $(\theta/2)$ – cos $(\theta/2)$ is positive if θ lies between $(\pi/2)$ (8m + 1) and $(\pi/2)$ (8m + 5) where m has any integral value.
>
> 8. Find the necessary relationship amongst the angles a, b, c that tan a tan b + tan b tan c + tan c tan a = 1
>
> 9. Give the complete trigonometrical solution of a triangle for which are given the base, the vertical angle, and the length of the bisector of the vertical angle between the vertex and the base.
>
> 10.Prove the identity 4 sin 2(a + b + c) sin (b – c) sin (c – a) sin (a – b) = sin (b – c) sin 3(b + c) + sin (c – a) sin 3(c + a) + sin (a – b) sin 3(a + b).
>
> 11. A triangle is formed by joining the centres of the escribed circles of a given triangle. Find an expression for the area of the former in terms of the sides of the latter.

In this paper at least, we see that trigonometry is entirely theoretical, a mixture of algebraic and geometric puzzles using trigonometric ideas and functions. One can sympathise with the examiners, to the extent that providing tables for students to be able to do practical problems must have been an impossible challenge.

It does appear, however, that the use of logarithms was at least being taught, because in May 1904 the examiners did manage to come up with a form of question that would test students' knowledge of the concept and its use:

> 9. Investigate a formula for the cosine of any angle of a plane triangle whose sides are given, and deduce a formula for the cosine of half the angle. The sides a, b, c of a triangle are proportional to 4, 5, 6 respectively. Find the angle B, given— log 2 = .3010300, L cos 27˚53' = 9.9464040, L cos 27˚54' = 9.9463371

[L cos is the then-standard notation for the logarithm of the cosine, subtracted from 10 to avoid negatives.]

Here they have found a way of testing the use of logarithms without having to provide tables, by essentially giving the requisite lines from a 7-figure table. In the process they have given an enormous hint as to where the answer lies, but as full working was an absolute requirement this would provide reassurance rather than giving the answer away. The working required is still intensely theoretical and detailed. There are many problems in similar format in Todhunter, the recommended text from the preceding century.

This was almost the last of the matriculation examinations before they were rendered obsolete by the new examination system of 1905; in those few papers of the old system that remained there were no further questions of this type.

The Public Examinations of 1905/6

After the turn of the century the pace of change in the upper secondary sector became more rapid, not least because in Victoria it became clear that the government was about to move for the first time into secondary education, hitherto the prerogative, jealously guarded, of the private sector. The University responded by recasting its matriculation examinations into a new form, a series of Public Examinations that covered all levels from the conclusion of primary education (the Primary Examination) through the Junior Public Examination and the Junior Commercial Examination to the equivalent of the modern year 12, the Senior Public Examination and its counterpart the Senior Commercial Examination.

In mathematics the Primary Examination (from 1905) included papers in Algebra, Arithmetic and Geometry. The Junior Public Examination (from 1905) and its companion Junior Commercial Examination (from 1906) shared many syllabuses, including the same Algebra, Arithmetic and Geometry; and from 1908 an additional Trigonometry paper was added. The Senior Public Examination was first offered in 1906, and included papers in Algebra, Geometry, Mechanics and Trigonometry, each offered at both Pass and Honours levels. Finally the Senior Commercial Examination included one mathematical subject, Commercial Arithmetic with Book-keeping, but this examination was rarely offered due to a lack of candidates.

All of these examinations in their mathematical sections make mention of the Mathematical Association report mentioned earlier, and seem to make a serious attempt to modernise their approach to curriculum. In particular the use of logarithms has become more widespread, at least as a senior topic. In the Junior Public Examination they appear only from 1908 in the Trigonometry syllabus:

> *Trigonometry*: A simple treatment of the following:— Angular measurements; addition formulae; relations between the sides and angles of a triangle; the use of logarithmic tables; solution of triangles.

However, in the Senior Public Examination they occur both in Algebra and in Trigonometry:

> *Algebra:* More advanced treatment of work prescribed for the Junior Public Examination together with the remainder theorem; quadratic equations of two unknown quantities; ratio; proportion; variation; permutations and combinations; binomial theorem for positive integral exponent; logarithms; the elements of the theory of partial fractions.
>
> *Trigonometry:* Angular measurements; addition formulae; relation between the sides and angles of a triangle; solution of triangles; easy examples in heights and distances; logarithms.

They also make a notable appearance in the Senior Commercial Examination syllabus:

> Commercial Arithmetic with Book-keeping:
>
> (A) Arithmetic: The work prescribed for the Junior Public Examination in Arithmetic, together with the following: Freights, rates of exchange and transactions with home and foreign Bills; the coinages, weights and measures of the principal countries of the world. Debentures, preference stock, ordinary stock, profits and dividends; liabilities, solvency and liquidation. Bankers' interest. The use of logarithms, more particularly for problems on compound interest, insurance and annuities. Methods of calculating rates and taxes. Compound interest with special reference to repayment of loans.

Logarithms did not appear as a technique in an Arithmetic subject until the Intermediate Examination of 1919.

Given the brief experiment of the May 1904 matriculation paper, how did the examiners of the new syllabuses manage to incorporate logarithms into their questions?

The first relevant papers of the new system were in the Senior Public Examinations of December 1906. The Pass Algebra paper includes the following:

> 8. Define a logarithm, and state and prove the rules for finding the logarithms of a product, quotient, and power.
>
> Find the number of digits in 2^{32}, having given $\log_{10}2 = 0.301030$.

This is ingenious, but the Pass paper in Trigonometry is more interesting. Of its ten questions, the first seven are theoretical questions mostly concerned with trigonometrical identities and deductions from them. The last three are rather more practical:

> 8. Shew how to solve a triangle having given two sides and the included angle.
>
> If $a = \sqrt{(3)} + 1$, $b = 2$, $C = 30°$, solve the triangle.
>
> 9. If a = 19.22, b = 23.04, A = 35˚12', find B and C, having given log 1.922 = 0.28375, log 2.304 = 0.36247, L sin 35˚12' = 9.76075, L sin 43˚42' = 9.83940; diff for 1' = 13
>
> 10. Shew how to find the height of an inaccessible tree. A vertical tower stands on a slope inclined to the horizontal at an angle of 15˚. At the foot of the slope the tower subtends an angle of 15˚. On walking 200 feet up the slope it is found to subtend an angle of 60˚. Find the height of the tower to the nearest tenth of a foot.

Apparently questions 8 and 10 were expected to be done using exact values and arithmetic, whereas question 9 is clearly an exercise in the use of logarithms. As in the 1904 instance, the information given allows an intelligent guess at the answer, but students would still have been required to write out full working in order to obtain credit.

In the corresponding Honours papers there is less practical use of logarithms. In the Honours Algebra paper the only relevant question is, not unnaturally, a piece of (difficult) algebra:

> 10. Define a logarithm, and find the relation between the logarithms of the same number to different bases. Eliminate x, y, z from $\log_{yz}x = a$, $\log_{xz}y = b$, $\log_{xy}z = c$.

The Honours Trigonometry paper is entirely theoretical.

In the following year the first Junior Public Examination paper in Trigonometry was set. The first six of its ten questions are either theoretical or involve calculations to be done without the use of tables (trigonometrical or logarithmic). But the final four questions all make use of the sort of information to be gained from tables, in various forms:

> 7. Prove that $a^2 = b^2 + c^2 - 2bc \cos A$ where A is the obtuse angle of the triangle ABC. If a = 4, b = 5, c = 8, find the angle C: given cos 54˚54' = .575.

A simple application of the cosine rule. The answer is (almost) obvious from the given information, but as usual working must be shown.

> 8. What is a logarithm? What are the numbers of which the logarithms are respectively 4 and –4?

A neat question that tests simple understanding.

> 9. Find cos 21˚27'45" if cos 21˚27' = .9307370 and cos 21˚28' = .9306306.

A simple exercise in interpolation, a necessary skill for users of tables.

> 10. Find the side AC of the triangle ABC from the following data:—

Angle A = 48˚, angle B = 54˚, side AB = 38 inches, log 38 = 1.5798, L sin 54˚ = 9.9079, L sin 78˚ = 9.9904; log 3.1430 = .4983.

Similar but not identical to the question from the Senior paper of the previous year, cited above, a piece of genuine calculation for which the answer is again "obvious" but working must be provided.

The contemporary Senior papers (December 1907) are very similar to those of the previous year; the most tabular question being the final question of the Senior Public Examination Pass paper in Trigonometry:

8. Shew how to solve a triangle when two sides and the angle opposite to one of them are given. If a = 3.9, b = 4.21, A = 62˚30', find the angles B and C. log 421 = 2.6243, log 3.9 = 0.5911, L sin 62˚30' = 9.9479, L sin 73˚12' = 9.9811. Comment on the double answers you obtain.

The 1908 papers are all remarkably similar in content and philosophy.

Change to the Provision of Tables

Over the next few years a major change takes place, in that at some stage the examining authorities began to provide complete tables with the examination, and expect that students would be able to use them in their work. This represents a major change in the mechanics of calculation, and in fact is the first substantial change in calculating technology in Australian senior secondary mathematics. Instead of the rather artificial questions we have seen so far in which the necessary tabular information is given and the answer effectively revealed in the process, we now see students being expected to display knowledge of the real-life procedures of calculating with tables, both trigonometric and logarithmic.

Unfortunately the actual timing and details of the innovation are not made clear in the official documentation: the sources which one would expect to announce and explain the change are completely silent on what is happening. In what follows I have simply made deductions from what can be seen in the actual published examinations and the examiners' reports.

In December 1909 the Junior Public Examination paper in Trigonometry contains two questions of the now-familiar type:

11. A flagstaff on a cliff is known to be 20 feet high. From a boat which is moored the flagstaff subtends an angle of 1˚30', and the elevation of the top of the cliff is 18˚. Find the height of the cliff and the distance of the boat from the base of the cliff, both to the nearest foot. Given— tan 19˚30' = .354, tan 18˚ = .325.

12. Two of the angles of a triangle are 48˚ and 54˚, and the length of the side adjacent to them is 38 in. Find the lengths of the other sides. log 38 = 1.57978, log 2.8870 = .46045, log 3.1429 = .49733, L sin 54˚ = 9.90795, L sin 78˚ = 9.99040, L sin 48˚ = 9.87107.

The Senior Public Examination Pass paper in Algebra contains a logarithm question, partly a written explanation and partly an ingenious calculation exercise:

10. Define the base of a system of logarithms. State what number is taken as the base of the common logarithms. Mention two properties of common logarithms which depend on the fact that that number is taken as their base.

Given log 2 = .3010300, log 3 = .4771213, log 2.94 = .4683473; find log 7.

It is in the Senior Public Examination Pass paper in Trigonometry that the first mystery occurs. There are two questions that require solution of triangles:

8. From the top of a hill the depression of a point on the horizontal plane below is 30˚, and from a spot three-quarters of the way down the depression is 15˚. If θ be the inclination of the hill shew that 3 cot θ = 3 √(3) – 2.

> 9. A person standing on the bank of a river observes the elevation of the top of a tree on the opposite bank to be 51˚, and when he retires 30 feet from the edge he finds the elevation to be 46˚. Find the breadth of the river.

In the absence of detail in the syllabuses or officially recommended texts it is hard to be sure what is intended here. In question 8 one suspects that students would be expected to know exact values for the sine and cosine of 30˚ and of 15˚ (as in Todhunter, 1869, p. 60), and hence work exactly using surd arithmetic: the form of the answer required makes this apparent. But question 9 is quite a puzzle: I cannot imagine anyone reaching a numerical answer without the use of tables (or a modern calculator). Were the examiners expecting an answer in exact unworked form, like (30 tan 46˚) / (tan 51˚ – tan 46˚) feet? The Examiners' Report (published in the Handbook of Public Examinations, 1911) is silent on these questions, making only the general observation that "This paper left a great deal to be desired, as in very few cases could it be said to be even tolerably done."

In the corresponding Honours Trigonometry paper there is only one question that raises similar queries:

> 9. Solve a triangle in which one angle at the base, the opposite side and the altitude are given; and explain when the solution is ambiguous or impossible.
>
> If the angle is 36˚, the opposite side 4 and the altitude √(5) – 1, solve the triangle.

Here the magnitudes given and the form in which they are given make it clear that exact working in surds is required, rather than the use of tables.

At the same time the Senior Commercial Examination was conducted for the first time, for a single candidate (who passed). The paper in Commercial Arithmetic and Bookkeeping marks the first occasion on which the use of tables was expressly permitted: at the top of the paper are the instructions:

> Candidates may use *Chambers's Mathematical Tables* for reference throughout this examination.
>
> Logarithms should be used wherever it is more convenient in working to do so.

Chambers' Tables being a large volume of over four hundred pages, one imagines the candidate provided his (or her) own copy.

Even more surprisingly, at the following Senior Public Examination in February 1910 some form of tables was provided with the examination paper in Pass Trigonometry. There is no information on this in the instructions on the paper, or in the "Materials for Examination" section of the Handbook of Public Examinations, which supposedly documents all official information about the system. But within the paper are a question (9) which cannot be done without tables, and another (7) on logarithms and their arithmetic which explicitly mentions tables:

> 7. Explain the properties of logarithms which enable them to be used to simplify calculations.
>
> Use the tables to find how many figures there there (sic) are in the ninth power of 2345, and find what £100 will amount to if left at compound interest for 100 years at 5 per cent.
>
> 9. Solve the triangle whose sides are 17.32, 15.89, 25.64 yards respectively.

We conclude that at some point about this time (1909 or 1910) a decision had been reached to provide some form of basic tables with the paper, and to examine their use in questions of more natural form than those previously possible.

It is unfortunate that there appears to be so little documentation of this change: together with the almost contemporary introduction of squared (graph) paper, it marks the first real introduction of mathematical technology into the school examination system since the beginning. The Handbooks do mention the issue of graph paper:

> Specially ruled paper will be supplied whenever the nature of the examination requires it ...

but say nothing at all about tables. The first mention in the official publications seems to be in the Examiners' Reports for 1911, published in the 1913 Handbook, which incorporates a rather unhappy statement about the use of tables in the Senior Public Examination Pass paper in Trigonometry:

> The solution of triangles was most unsatisfactory. Though most candidates had an adequate knowledge of the necessary methods, they failed signally in obtaining accurate results. The causes for this failure usually were most inexcusable, being such as misreading the mathematical tables, interpolating incorrectly, using the wrong table, etc. The mathematical tables supplied for the examination are not arranged in any unusual form, and should be readily understood by any candidate, if not already quite familiar to him. It is not demanding much that exercises in the solution of triangles should be accurately worked with four-figure logarithms by the candidates for the Senior Examination. ...

From this we can conclude that, while the tables were of a standard form, there is some admission that not all candidates might have been familiar with them prior to the examination. We also note, significantly, that the tables were worked to four figures.

In the absence of documentation we can at least speculate on what form the tables took, and why some candidates might not have been familiar with their structure. As mentioned before, the standard tables of the time were *Chambers's Tables*, a large volume of seven-figure tables that was clearly unsuitable for student use (Pryde, 1878). The breakthrough into student and school practice seems to have come with the argument of Comrie (1972) and others that at the school level the use of four-figure tables would provide sufficient accuracy for most purposes, and more than the conventional slide rule. The essential difference is that whereas a 7-figure table of logarithms takes about 200 pages of conventional type, and a 6-figure table about 50 or 60 pages, the standard 4-figure table of logarithms is only one double page, and thus easier to use and certainly easier for a school or an examination authority to supply to all students.

It would be more than interesting to discover some background material about the circumstances of this change. It may well be that the authorities in Melbourne were in the vanguard of educational practice in this respect, because the commercially published books of four-figure tables that were a familiar feature of school mathematics until the 1970s were all first published in the 1920s, or later. One suspects that the tables provided by the examiners of 1910 were home-grown affairs, possibly compiled at the University expressly to accompany the Public Examinations: if so, it was an innovation of great foresight and its originators should be honoured as pioneers in the ever-accelerating advance of calculating technologies.

References

Barling, C. R. (in preparation: 2003a). Multiplication Before Calculators.

Barling, C. R. (in preparation: 2003b). Some Nineteenth Century Table Books.

Blake, L.J., ed (1973). *Vision and Realisation: A Centenary History of State Education in Victoria*, 3vv. Education Department of Victoria.

Comrie, L. J. (1972). Preface to *Chambers' Shorter Six-Figure Mathematical Tables*. Edinburgh and London: W. & R. Chambers Ltd.

Griffiths, H. B. and A. G. Howson (1974). *Mathematics: Society and Curricula*. Cambridge, Cambridge University Press.

Pryde, J., ed. (1878). *Mathematical Tables*. London: W. & R. Chambers Ltd. ("New edition", n.d.).

Sweetman, Edward, ed (1922). *A History of State Education in Victoria*. Education Department of Victoria.

Todhunter, I. (1869). *Plane Trigonometry for the Use of Colleges and Schools*. Fourth Edition. London and Cambridge: Macmillan & Co.

University of Melbourne Calendars, 1858/9 –

University of Melbourne: Melbourne University Examinations, 1899–

University of Melbourne: Handbook of Public Examinations, 1904-