# Opportunities to Learn Mathematics

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Perception of the mathematics classroom as an arena in which there are various opportunities to learn mathematics leads to a fine-grained focus on the structure of mathematical tasks. Each mathematical task affords engagement with mathematics in certain ways. Variation within a task is a major factor influencing learning.

I was teaching a year 9 (13 year-olds) all-attainment[1] class. They had been working on this task:

> On a coordinate grid you are only allowed to move to the right or upwards. You can do this in any order you like. How may routes are there from the origin to the point (1,1)? How many routes to the points (1,2), (2,1), (1,3) , and so on…..?

After about ten minutes I gathered all the students around the board and asked them what they had found. Silently at the back of the group sat Paul, who had been described to me as having special needs. When he had entered school he could not talk, and still at 13 he could neither read nor write. After several students had described how they had counted routes systematically, and deduced a sort of symmetry emerging, I challenged them to find a method which allowed them to work out how many routes there would be to get to any point, for example (6,7). Paul said immediately 'if I knew how many it would take to get to (5,7) and (6,6) I could add them to get (6,7)'. This reply would have been a pleasant surprise from any student, but from Paul it was doubly so because it was his first utterance in such a group. This was a turning point for me as a teacher, and for him as a learner. My expectations of his mathematics were biased by what I had been told, and he was able to grasp spatial situations with an abnormally skilful level of generality and structure.[2] I would have to work on my expectations and Paul would have to work on mathematics through spatial representations.

Shortly after this lesson I began to wonder about how teachers make judgments about their pupils' mathematics, and did some research about this (Watson, 1999). I found that teachers seemed to have no qualms about saying that their judgments came from 'knowing the child' or 'gut feeling' or 'professional judgment'. The latter phrase seemed to mean 'experienced judgment'. One of the 30 teachers I interviewed noticed, as she was talking to me, that she had inadequate strategies in place to ensure her judgments were fair. She realised that she depended on seeing certain facial expressions, but that many of her students, being young Muslim women, would keep their faces down and she would never see their expressions. As a result of this research (of which the observations above are only a glimpse) I drew the sad conclusion that, even with teachers' judgments contributing towards high-stakes assessment decisions, there was a lack of serious monitoring and professionalism about their impressions of students.

## Construction of Types of Learner

This was not new of course. If there is a label around, such as ‘low attainer’, then teachers, schools, classmates and even parents will act together to apply that label to particular individuals (McDermott, 1993). The existence of the label acquires people to fit it. Similarly, the label ‘high achiever’ will cast around looking for students to fit. Nash (1976), Blease (1983), and others found that teacher expectations create self-fulfilling prophecies.

Atweh and Cooper (1995) show clearly how that can happen, by demonstrating how gendering of mathematics achievement can be created by many aspects of classroom practice. In a girls’ school, a teacher acted as if mathematics was an irritation to be tolerated within a general discourse of social events and relationships. Yet she retained the role of mathematics authority, being in sole charge of mathematical warrants. The students in her class had plenty of opportunity to learn that mathematics was not really important, was rather random, and was only understandable by experts. They had few opportunities to learn the extent of their own capabilities for acting mathematically.

The students could, of course, have rebelled and set up their own alternative discourse. In many classrooms there are some mutterers who may be rejecting the curriculum, but who may merely be setting up an alternative way to see mathematics and have not been enculturated to behave in the way the teacher intends. Houssart (2001) identifies a group of year 6 boys who have a private enquiry system going on at the back of a classroom, in which they answer the teacher’s questions rapidly *sotto voce,* write little, and call out challenges to the teacher. Needless to say, they are on the edge of becoming alienated by school mathematics.

I find Zevenbergen’s analysis of the construction of social difference in mathematics classrooms which theorises these phenomena in terms of linguistic capital quite persuasive (e.g. 1998). To appeal to the teacher a student has to use certain forms of language and behaviour, and she can only use appropriate forms of classroom discourse if she has had opportunity to acquire them, and encouragement to use them. Once having used them, a student needs to have that use recognised and validated. At any stage of this process teacher preference and bias can come into play to distinguish, fairly or not, between those that ‘fit’ and those that do not. Norton, McRobbie and Cooper (2002) write about five teachers who use explanatory or investigative methods with ‘able’ students and ‘show and tell’ with ‘less able’, presumably because the ‘less able’ are less able to convince the teacher that they can participate in more sophisticated forms of mathematical interaction. Those who fail to adopt such forms in early opportunities therefore get fewer opportunities to exercise them subsequently.

Wherever students are segregated according to some notion of ability for mathematics there are similar constructions at work. The teacher constructs some learners as deficient and offers fewer opportunities to learn. When teachers use the phrase ‘less able’ one might ask what exactly it is that these students are less able to do. Often what is seen as lacking is a skill or behaviour which can be taught, learnt, or encouraged, or which the student already has in some other arena but has not used in mathematics. Opportunities to learn new behaviours, or to use what one already knows in other contexts, can open up opportunities to learn mathematics. In classrooms such as those discussed above, opportunity to learn is limited by the teachers' construction of mathematics, of mathematics discourse, and of learner-types.

## Opportunities in Classrooms

Social, cultural, and linguistic lenses are useful in answering the question 'what do learners have the opportunity to learn?' because part of what they learn is what is acceptable in mathematics lessons, both in terms of behaviour and also in terms of their relationship with mathematics. These are becoming well-worked seams in mathematics education and appear to explain the behaviour of certain groups, certain classrooms and some individual responses. Do students act out what the teacher and textbook say, and have this validated by the same authorities? Or can students participate in the construction and validation of mathematical meaning, and what would this mean?

### *Individual Learners as Negotiators of Meaning*

A common view of learning mathematics is that meaning should be negotiated. I want to distinguish between meaning being *negotiable* and meaning being *negotiated*. Mathematics is not negotiable because it has to have internal coherence and validation; however mathematics comes into being it has to be a system with these characteristics. Anyone can validate a statement as mathematical, but they have to show it is valid within the terms of mathematics.

If we want to claim, as some argumentative people do, that 2 and 2 could make 5 some day, then we cannot use the word 'plus', we would have to invent a new word. It would be entirely possible to define an arithmetic in which $2 * 2 = 5$, but we would need other examples in order to grasp its structure. And it might be quite fun to work out what else would have to be true, and then to test various numbers and operations and features of the structure which then unfolded. Negotiated meaning and negotiation are legitimate features of maths classrooms. It is not too fanciful to imagine a classroom in which students are constructing this new arithmetic with students and teachers providing test cases and counterexamples and making up new language to describe what is found. But the role of the test cases and counter-examples is not negotiable because an essential feature of mathematics is its own internal validity.[3]

Similarly the derivative of $x$-squared with respect to $x$ is $2x$ … nothing negotiable here either. Yet we may have to negotiate with students who think it is $2x + k$, (as indeed some do), starting from the recognition that something is being thought – but what? How would we continue with such a negotiation? A teacher could then ask for the derivative of $x^2+kx$ with respect to $x$? The presentation of a possible case for conflict is much more than an attempt to enculturate the student into conventional understandings, or to extend the student's experience of a particular discourse, it is an induction into an internal validity of mathematics by offering the student an opportunity to rethink and restructure existing assumptions and understandings, or at the very least to realise that there is a problem. By offering a very particular example the teacher informs the negotiation not from finding out what the student understands, not from making a judgment about what the student can handle, but by giving the learner the responsibility to sort it out for herself, given enough information and experience. The learner has an opportunity to learn, or at least to recognise that something has to be thought about some more.

The three dimensions of mathematics teaching identified by Holton and Thomas (2001, p.87), cognitive, metacognitive, and affective, can all be seen as manipulable in order to provide different opportunities. Lessons which explicitly pivot around negotiation do not only attend to the metacognitive and affective aspects of mathematics lessons but also can be organised to be cognitively powerful. In classrooms where students actively negotiate meaning, the teacher's role is to structure the content of such negotiation with examples, counter-examples, or by encouraging the development of these, so that what is eventually learnt is coherent and valid, but also to structure the negotiation process itself so that it is mathematical, by which I mean that it is based on exemplification, generalisation, conjecture, justification, and so on.

### *Cognitive Opportunity*

By focusing on cognitive opportunities to learn I look at students as being similar, rather than different, because as much can be gained from assuming similarity as by trying to explain, find out about, understand and take account of difference.

In one of the lessons described in Atweh and Cooper (1995 p.302), students were presented with an equation ($k + 5 = 9$) and told that the purpose was to find the value of $k$.

The teacher states that 'k is a mystery number' and asks 'who knows what to do?' The learner, on the other hand, looks at the example and knows that k has to be 4, it is not a mystery. The learner can also can give a reason like 'because 4 plus 5 is 9' and therefore has no idea what 'who knows what to do?' can possibly mean. The teacher probably wants students to construct a method, a personal set of instructions, a way of seeing a solution which will work both for similar cases, and more complex cases. For the learner, this example offers little opportunity to learn meaningful mathematics because there is no reason to engage with what the teacher is doing; there is nothing puzzling about the example. It is not just that the representation is algebraic, (although that would alienate students like Paul), or that the teacher has not used forms of interaction which engage all students, or even that this equation appears like magic from nowhere – it is that there is nothing to learn for *any* student except those who take the way the teacher sees mathematics on trust, and trust in this classroom was fortunately abundant. The teacher, by creating a participative atmosphere, overcame the potential meaninglessness of the task. However, the class consisted of students who were aspiring to university, so were already attuned to suspending everyday knowledge in order to function in school.

The teacher thought the task was to learn a method to find the value of $k$, but, as Christiansen and Walther (1986) point out, the activity of the learner is not necessarily the same as the task the teacher imagines setting.

Contrast the example above to this sequence:

$k + 7 = 11$

$k + 6 = 10$

$k + 5 = 9$

The sequence draws attention to some changes; the only thing which stays the same is the letter $k$. It stays the same in a rather dominant way, being at the front of each line. Thus the changes in the numbers seem obviously not random. A large majority of people would be able to say what the next line could be, and what the previous line could be, in order to retain the pattern. Extension of this sequence is this manner could lead to $k + 0 = 4$ or onwards to negative numbers. If attention shifts from going *with* the grain of this list, up and down, to looking *across* the grain at underlying structure, it would be a small shift to produce $k + n = n + 4$. The value of $k$ is still not a mystery, so there is still no reason to learn how to find $k = 4$, but by the nature of changes and constraints there are things to learn about structures relating to the statement "$k = 4$". Students could produce their own similar sequences related to "$k = 3$". It may even be possible to set up a new sequence based on something harder, such as $m - 7 = 156$, and devise instructions to solve it based on the experience of working with similar structures in an exploratory way. The constraints on variety in this approach can lead to powerful recognition of patterns which easily become rules for engagement with other examples.

But to be courageous enough to try this approach instead of 'show and tell', or even instead of 'explain and do', we have to believe something about learners' similarities. This example depends on the general propensity to spot and use patterns, and to be able to say something about habits, using the patterns as raw material for mathematics, and thus to make some form of general statement, which someone, the teacher perhaps, then expresses in symbols. Furthermore, what is written is a spatial pattern of symbols, not words, so is more, rather than less, accessible to language-deprived students. What is written on the board does not require commentary, so anyone who has been daydreaming can re-enter the interaction and make some sense of what has gone on. The entire problem can also be cast as a sequence of graphical or diagrammatic representations so that a student like Paul can have access to it, and other students can relate the different representations to each other at an extra level of understanding.

Consider this exercise which is supposed to be about ratio and was set by a teacher in one of David Clarke's classroom videos (see Lerman, 2001):

1. Reduce to simplest term

(a) $\frac{4}{12}$

(b) $\frac{36}{12}$

(c) $\frac{240}{300}$

(d) $5 : 5$

(e) $ab : ab$

(g) $2\frac{1}{4} : 1$

(h) $\frac{6ab}{3b}$

There are so many different things one could focus on here that it is tempting to give up trying to make sense of the variety and say 'every question has pairs of numbers to cancel down' without any underlying sense of meaning. In fact, this becomes an exercise in doing cancelling, not for learning more about ratio. In order to learn something about ratio, one could open the work up for whole class discussion and ask what are the similarities and differences between the questions, or what sort of situations would lead to such statements, or how one could represent the different statements. As it stands, this exercise has the potential to *exacerbate differences* in learning. There is nothing wrong with setting tasks which have a variety of responses; such tasks can promote participation, reduce risk, and encourage exploration. But the demands and effects of testing make this exercise an arena for potential exclusion of some students. They are supposed to learn something about ratio from it - but what? There are too many things to learn and none of them are developed.

Opening the task up through whole class discussion of the meaning of the various types of ratio would certainly produce clearer opportunities to learn, but so also does closing it down even more.

Consider these examples of relationships:

21 : 21; 6 : 6; 35 : 35

In these three ratios there is something trivial to notice, which is that in each example the numbers are equal, but they are also examples of a generalisation which appeared as (e) in the exercise above, $ab : ab$. By offering a list of such pairs, learners are being encouraged to construct for themselves the generality that $ab : ab$ reduces to 1:1, and they can do that either by dividing by $a$ then $b$, or $b$ then $a$, or by $ab$. Suddenly an apparently trivial question becomes an arena for generalisation and for an articulation of a result which depends on the Fundamental Theorem of Arithmetic.[4] The exercise becomes much richer than before, still gives practice in cancelling, and promotes mathematical thinking as a bonus. Other sequences might extend students' understanding of ratio structures towards some complexity, such as: $ab:a^2b; pn:pm; pk/pl: pm/pn$ , and so on.

All tasks offer cognitive opportunity, we know that learners construct their own meanings whatever is offered, but tasks can be structured so that useful generalisation is more likely – there is more opportunity to make mathematical meaning. Once this is done, differences in learning due to other factors begin to fall away because all that has to be discerned is sameness and difference.

### *What Is There to Generalise?*

If I see things as very different my tendency is to categorise, not to generalise. If I am forced to generalise then I may do so wildly, unhelpfully or inappropriately. For example, imagine being asked to generalise a blue bicycle, a blue teddy bear, and a pair of blue gloves. Perhaps 'blue' is the generalisation required, perhaps they all belong to one person, perhaps they are all lost property … who knows? If I do not know why I am being asked to generalise, how should I proceed? Another problem occurs when there is so much variation that students generalise differently. Imagine a supermarket with the goods categorised by colour because that is the most immediately obvious feature to someone, or by package size because that is most obvious for someone else. Similarly, what seems to a teacher to be a set of examples about ratio appears to the learner to be a set of examples about factorisation or cancelling. We naturally generalise when things seem to have a strong connection, not when they seem to us to be very different. If things are very different, we naturally tend to categorise rather than generalise, just as we do when learning and using language. So by offering a wide variety to students we exacerbate differences since they will want to categorise rather than generalise, and they will choose from a range of foci according to previous experience, knowledge, mood and immediate perception. If we offer little variety it is more likely they will make similar generalisations. If we offer no variety there is nothing to notice, nothing to learn. The similarity among learners on which I base these observations is their similar propensities to look for pattern, to generalise, and to categorise. Opportunity to learn is focused and refined by controlling the variation in what is offered so that useful generalisation is likely to arise from students' normal thinking processes.

The organisation of the school curriculum requires that the more complex the mathematics gets, the faster we expect students to generalise. They generalise addition and subtraction over a couple of years, but we expect them to generalise about trigonometry over a couple of weeks.

### *Learners Perceiving Opportunities for Choice*

Analysing mathematical tasks in terms of the dimensions of variation they offer learners can explain some aspects of students' behaviour. For example, an experienced teacher, Sara, had chosen a tasksheet from a published collection to help her low-achieving students engage with algebraic representation of patterns. Students were instructed to create a repeating pattern by colouring a line of squares and then to describe their pattern by substituting letters for the colours: *a,a,b,a,a,b* … , and so on. To her frustration, the majority of students spent the lesson choosing their colours and colouring squares very carefully.

There are several ways to explain this unsurprising response. You could say the students had been used to being given meaningless mundane tasks by their previous teacher, so repeated old habits, or that the algebra was rather frightening and they were avoiding it, or that they were exhibiting learned helplessness in the face of a challenge (Peterson, Maier, & Seligman, 1995). All of these might be true. Analysis of the dimensions of variation was enlightening, because the first statement on the tasksheet was 'choose two colours'. Since this was the first choice offered, and the first dimension of variation, this was the one they focused on, for whatever reason. To get them to focus on algebra, it should have been the algebraic representations they were invited to choose and vary. A restructuring of the task to offer different algebraic patterns as a first dimension of variation, worked much better:

> Using $p$ and $q$, make a repeating pattern of letters in a line. Let $p$ stand for white and $q$ for black and represent your pattern as a line of squares…… Do at least 5 patterns.

This formulation forces students to make the letter pattern immediately, and to use the colours (without choice) later to illustrate it. The pattern becomes the generator for something more visual and attractive, rather than the reverse. Students in another class who were offered this reformulated task spent more time working with letters, where the choices they made related directly to the mathematics of the task, and much less time colouring.[5] Opportunity to spend time on irrelevant variations was reduced.

## Dimensions of Variation

My development of the use of *dimension of variation* as a tool for analysing both tasks and student responses owes much to others. An early version of the English National Curriculum for Mathematics included some Non-Statutory Guidance (NNC, 1989) which gave advice for turning closed questions into open questions, and Sullivan and Clarke (1991) provide some further ideas and theoretical support for this kind of approach. More recently, Prestage and Perks (2001) have developed a wide range of generic methods for adapting tasks in an effort to help teachers use textbooks imaginatively. While writing *Questions and Prompts* I became dissatisfied with the common distinction between open and closed questions (Watson & Mason, 1998). It became clear that what was important about tasks was not the openness of the question, but the opportunities it offered the learner to adapt, extend, and refine a personal understanding of the concept. Sometimes a closed question would do this better than an open one, and this could depend on many cognitive, metacognitive, affective, social, and cultural factors. Often the nature of mathematical relationships directs the choice. In an example given above, the closed question 'what is the derivative of … ?' is helpful in indicating a contradiction without having to tell the student they are wrong; the authority rests within the mathematics.

Marton sees learning as the discernment of variation in events which occur almost simultaneously and Marton and Booth (1997) use dimensions of variation as a way of looking at the different learning outcomes of similar teaching situations. Runesson (2001) has developed this view to explain differences in mathematics lessons which, while being planned together, turned out to be very different lessons in practice. Two teachers co-planned a lesson on fractions of a number; in practice one teacher offered students various fractions of various quantities while the other offered the same fractions of various quantities. The first teacher was offering several dimensions of variation, while the second was offering one, namely the size of the 'unit'. We have extended this idea elsewhere to recognise that learners constructing their own examples have their own perceptions of what dimensions of variation are possible for a concept, and what range of change they allow themselves along each dimension (Watson & Mason, forthcoming).

It may be that some mathematical topics lend themselves to this kind of analysis more easily than others. Groves and Doig (2002) offer videos of two lessons in which there are many contrasts. In one lesson, students are arranged to stand along a straight line and throw rings over a pole a little distance away from the line. The class then discusses how easy/hard it is to do this, and distances are 'measured' and compared using tape. They conjecture that the way to make the game fair would be to stand in a curve which turns out to be a circle. In the other lesson, students are tossing three coins and recording their results, and they end up having a rather inconclusive discussion about the different results they found. There were many features in these lessons which could be contrasted, but the one which stands out for me is the different nature of the tasks. In the first, the dimension of variation was the distance from the point. This was enacted by students, represented by string, and made the focus for discussion. No one in the lesson, or observing, could fail to notice that the lesson was about distance from the point – there was little else going on. In the second task, there were variations in numbers of trials, methods of throwing, ways of recording, and ways of comparing results. Any teacher observing this would have known exactly that the lesson was about probability, but for many students it might have been mystifying.

I used to claim that the variety of responses and kinds of learning which would take place in a lesson such as the second were valuable (Watson, 1988). Some students would be learning about ways of recording, others about fractions, others about how to work together, others about how to avoid working, and so on. Thus each student is free to construct their own understanding of the lesson. Then I realised that without intervention some might only ever learn about ways of recording, and some might only ever learn about ways to work together, and only a few might learn about experimental probability. My role as a teacher was to narrow the range of what it was possible to learn, to discern as varying, in what I offered disparate students and thus increase the opportunity to learn appropriate mathematics for as many students as possible, while making sure that they all had access to the patterns under consideration.

Identification of dimensions of variation is a tool which extends to every level of classroom practice. I can use it to question the effectiveness of the ratio exercise, or the algebra task, or the throwing lesson, or the probability lesson, and I can use it to explain what students are discerning when they respond in unexpected and unintended ways. I can use it to theorise similarities in learning behaviour without having to adopt purely cognitivist or behaviourist approaches. Cultural and historical differences may lead learners to see different dimensions of variation when there are several around, and to make different extensions from those they see, but the fewer there are, the more likely it is that they will notice the same variations, even if what they do as a result varies. Thus good tasks can be planned by starting from mathematical structure, growing outwards, as it were, to recognise and engage cultural diversity.

## Affordances and Constraints

I shall now return to a socio-cultural perspective in order to see how this approach fits with the idea of a classroom as an arena in which the identities of learners develop. Greeno (1994; 1998) applies Gibson's articulation of social settings as structured by affordances, constraints, and attunements. Classrooms have the potential to be places of the exercise of power, of language, of personal success or failure, of working alone or with others, of working on mathematics or working on carving initials on a desk. These affordances are constrained in some way, by tools, rules, custom, language, power, so that the actual possibilities are a subset of what might be possible. Each individual brings their own attunements into this arena which influence perception, recognition, responses, and thus affect the whole setting. There is a complex interplay between what *could be* possible, what *is* possible, and what *is seen as* possible.

This analytical frame can operate at several levels. It offers a lens with which to examine institutions, classrooms, individual lessons, and one-to-one interactions. It also offers a lens through which to view the nature of tasks and the activity which ensues. A mathematical topic provides an arena which affords learning, constraints limit the variation which can be perceived, and learners bring attunements which include their capacity to categorise and generalise.

Thus the design of the mathematical task is a crucial factor in providing opportunity to learn. By limiting variation to the feature on which we hope students will focus, and inviting conjecture and generalisation, we can construct (and reconstruct) learners as mathematical thinkers by using the thinking skills they naturally possess to construct mathematical meaning.

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[1] I use this phrase instead of 'mixed-ability' in order to avoid the notion of 'ability' as being fixed.

[2] I have since learnt that the ability to grasp spatial structures rapidly as complete objects is a special strength of some dyslexic people (ref. Gift of Dyslexia..ref)

[3] I am aware that some philosophers could question this. Nevertheless we behave, and expect students to behave, 'as if' it has internal validity, and I would rather the authority rested within a shared understanding of mathematical structure than in the offices of the examination authorities. If there were no internal validity I find it hard to see what place mathematics could have in education.

[4] The Fundamental Theorem of Arithmetic states that there is a unique prime factorisation of every number. Thus it follows that one can divide it exactly by a sequence of its prime factors in any order until you reach unity. If the prime factorisation were not unique, then dividing by *a* may not leave you with something you can divide by *b,* and *vice versa.*

[5] Whether they were actually doing algebra is another question, because this formulation clearly exposes the fact that the letters here do not represent numbers.