Potential of Technology and a Familiar Context to Enhance Students’ Concept of Rate of Change

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Students’ concept image of rate of change may be incomplete or erroneous This paper reports a pilot study, with secondary school students, which explores the potential of technology (JavaMathWorlds), depicting a familiar context of motion, to develop students’ existing schema of informal understandings of rate of change to more formal mathematical representations Students developed numerous ‘models of’ rate of change in a motion context which then transferred to serve as a ‘model for’ rate of change in other contexts

This paper explores the potential of technology to facilitate the forging of connections between students’ prior informal comprehension and formal mathematical concepts of rate of change in order to build a more robust concept image Martin (1994), reporting on an investigation of the use of graphing technologies to improve the conceptual understanding of pre-calculus concepts, states that “technologies offer many potential benefits that could help to improve conceptual learning and mathematical power that last into subsequent courses” (p 170)

Encouraging students to develop more than a superficial understanding of rate of change is often difficult (Orton, 1984; Stump, 2001) Students may develop an algebraic manipulative competency but still not understand the basic concepts underlying the processes they perform Orton claims that students were “not at all happy with rate of change in the context of linear graphs” and had a “limited grasp of rate of change” for non-linear functions He proposes the use of real-life situations to stimulate meaningful discussion Stump also reports that “many students [in her study] had trouble interpreting slope as a measure of rate of change”(p 81)

For many Australian students their first formal mathematical encounter with rate of change arises in a study of linear functions with an emphasis on the gradient of these lines The focus is commonly a decontextualised calculation: substituting values (the co-ordinates of two points on the straight line) into a formula Sometimes the result of the calculation is then interpreted in terms of the steepness of the graph of the function Hauger (1997) suggests that the emphasis, in schools, on separate straight lines and their properties limits students’ understanding of rate of change to the slope of a straight line

Emphasis on calculating gradient using a rule which returns a single value applicable across the full domain of a linear function may limit students’ understanding of the concept of gradient Indeed, Stroup (2002) refers to “the extraordinary difficulty the vast majority of students have with the traditional approach of trying to “build up” complexity from the simplicity of the linear function (constant rate)” (p 206) He asserts that the linear function is “in the way” and advocates that learners should start with situations where rate varies The traditional approach to the derivative of a function leaves a discussion of the gradient of a curved graph until after the formal introduction This approach may overload the students’ cognitive resources and cause confusion (Tall, 1985) Tall proposes that an overview of the idea of the gradient of a general function graph may be explored using a calculator or graphing software He advocates that students experience the changing gradient of a curve by considering sections which are 'locally straight' seen by zooming-in on a calculator or graphing software

These concerns and ideas suggest that an investigation of an alternate approach to developing a more complete and useful concept of rate of change using piece-wise linear functions was warranted The aim of the pilot study incorporating the use of technology, reported in this paper was to explore the suitability of 'velocity' as a model to encourage rate-related reasoning (Stroup, 2002), for middle-years secondary students, to avoid the constrained thinking resulting from the introduction to gradient via the formula Also intended was a trial of classroom materials and data collection instruments

This paper draws on data from a classroom study using the animation software JavaMathWorlds (MathWorlds) (SimCalc Project, n d) This software "provides dynamic, direct manipulation graphs, piecewise definable functions, and animated cartoon worlds to engage elementary, middle, and high school students in qualitative and quantitative reasoning about the relationships among position, velocity, and acceleration in complex contexts" (Roschelle, Kaput & Stroup, 2000, p 47) The use of piece-wise linear functions in this environment is consistent with Tall's (1985) advice that students experience the changing gradient of a curve by considering straight sections MathWorlds affords the solution of problems involving multiple linear segments These portions of piece-wise linear functions allow students to observe the effect of changes in gradient

The following sections will cover: some theoretical aspects, including connection of informal to formal mathematical constructs, technology-rich learning environments, cognitive residue, multiple representations, and emergent models; a description of a pilot program comprising the software and supporting materials; data collection methods; and finally some results and their possible implications

Background

In order to begin with a shared meaning of the terminology, the following sections set out the theoretical perspectives which have informed this study

Previous Knowledge

Students bring with them a wealth of experience with rate of change in every day contexts but often have difficulty connecting their informal understandings of rate of change with the more formal mathematical approach Sfard (1991) advocates that connecting new knowledge to existing schema was more likely to result in long term retention Similarly, Hiebert and Carpenter (1992) espouse the view that stronger and more numerous connections within and between networks of internal representations aided memory and recall It seemed reasonable, then, to expect that connecting formal mathematical representations of rate of change to students' informal understanding of velocity could result in increased or deeper understanding of rate of change, independent of the use of a formula to calculate gradient, and that the parameters of linear functions might be associated with real meaning outside the classroom Tall and Vinner (1981) define concept image to be "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (p 152) The development of a robust concept image can be assisted by providing many different experiences related to a particular concept For example, students' concept of functions can be enhanced by viewing functions graphically, numerically and symbolically

Multiple Representations

Learners perceive aspects of the world around them and interpret them by attempting to fit them into their existing schema (Sfard, 1991) of knowledge and experience Providing multiple formal mathematical representations of a concept such as rate of change allows students to view the concept from their preferred representation (Keller & Hirsch, 1998) facilitating sense-making and ease of connection to their existing schema Similarly, Kaput and Schorr (2002) emphasise the importance of linking the numeric, graphical and symbolic representations of a function to a simulated situation in order to deepen qualitative understanding of the concept of rate of change

Technology-rich Learning Environments

Technology is providing ease of access to several representational modes linking real-world contexts to graphical, tabular and symbolic representations (Kaput, Noss, & Hoyles, 2001) enabling students to connect their existing schema of informal understandings to the more formal mathematical representations of a concept In this study, MathWorlds is examined as a vehicle for providing an enduring bridge between the informal and formal, leading to a concept image of rate of change which is mathematically correct and potentially useful for further study in calculus

Cognitive Residue

Such a bridge may be seen as part of the cognitive residue of this instruction Many writers (Pea, 1997; Salomon, Perkins & Globerson, 1991) in the fields of Computer Supported Collaborative Learning (CSCL) and Computer Based Learning (CBL) use the term 'cognitive residue' to refer to the after effects of instructional sequences based on the use of computers and other technologies such as graphing calculators and more recently CAS calculators Their use of the term implies more than just the learning of knowledge and skills, inferring that strategies and general approaches are an effect of exposure to a technology-enriched instructional sequence Salomon et al suggest that "higher order thinking skills that are either activated during an activity with an intellectual tool or are explicitly modeled by it can develop and can be transferred to other dissimilar or at least similar situations" (p 6)

Emergent Models

A parallel to the idea of an enduring bridge between the informal and formal mathematical representations as the cognitive residue of technology-rich learning environment may be drawn with Gravemeijer's (1999) emergent model theory The emergent model is the final result of a salodynamic mathematical modeling process in which, over time, the situated, informal models (*model of*'), developed by students to solve problems directly and uniquely related to specific situations, are transformed into a more general model (*model for)* used to solve problems in other different, but similar contexts So, for the student, a new mathematical reality is created

Consequently, in this study, we are looking for evidence of an emergent model for rate of change in non-motion contexts as cognitive residue of students' experience with technology-supported multiple representations of piece-wise linear functions in the context of speed

Methodology

The participants in this pilot study were a group of twelve Year 10 (16 year olds) students at an Australian secondary school Classroom materials, assuming the use of animation software MathWorlds, were based on those obtained from the SimCalc website (SimCalc Project, n d) but, in consultation with the class teachers, were adapted to suit the requirements and limitations of this teaching environment

MathWorlds simulates the movement, for example, of a lift in a multi-storey building This context is experientially real for students linking to their prior knowledge in the manner suggested above (Hiebert & Carpenter, 1992; Sfard, 1991) and manipulation of representations for a limited number of similar contexts may build 'mental pictures' (Tall & Vinner, 1981) or 'models of' (Gravemeijer, 1999) these particular contexts, whilst at the same time creating 'models for' other situations

The movement of the lift is also represented by a position-time graph, a velocity-time graph, a numeric display, a table, and an algebraic rule (Figure 1) A powerful feature of MathWorlds is the interconnectivity of control between the multiple representations of the movement of a lift Changes in one representation of the simulation are mirrored in the other representations For example, changing the symbolic representation of the situation results in corresponding changes in the animation, graphs and table The movement of the lift is controlled by its representations and the representations of its movement are controlled by the animation For example, changing the gradient and translating the graph automatically changes: the speed of the movement of the lift and the starting point for its journey; the values in the table; and the parameters of the symbolic representation

The first named author, working in close consultation with two class teachers, considered vocabulary, suitable scenarios, and previous experience with mathematical concepts and associated language The context of a moving lift was favoured to highlight the connection between the vertical axis of the position-time graph and the depiction of the animation beside it Care was taken to ensure that the scaling and location of the vertical axis of the position-time graph matched the floors of the stylized building in which the lift moved Four lessons were prepared The first lesson focused on the use of MathWorlds to represent motion of lifts with position/time graphs and tables; the second explored linking the symbolic and graphic representations in the MathWorlds environment; the third investigated average rate of change and, the final lesson considered rate of change in some different motion contexts Due to timetable constraints, these lessons, including adequate time for pre and post-tests, were trialled during three two-hour sessions over a four week period Students' scripts from pre and post-tests were examined and three teacher-selected students were interviewed immediately after the post-test regarding their written responses The researcher and class teacher discussed the progress of the class after each session

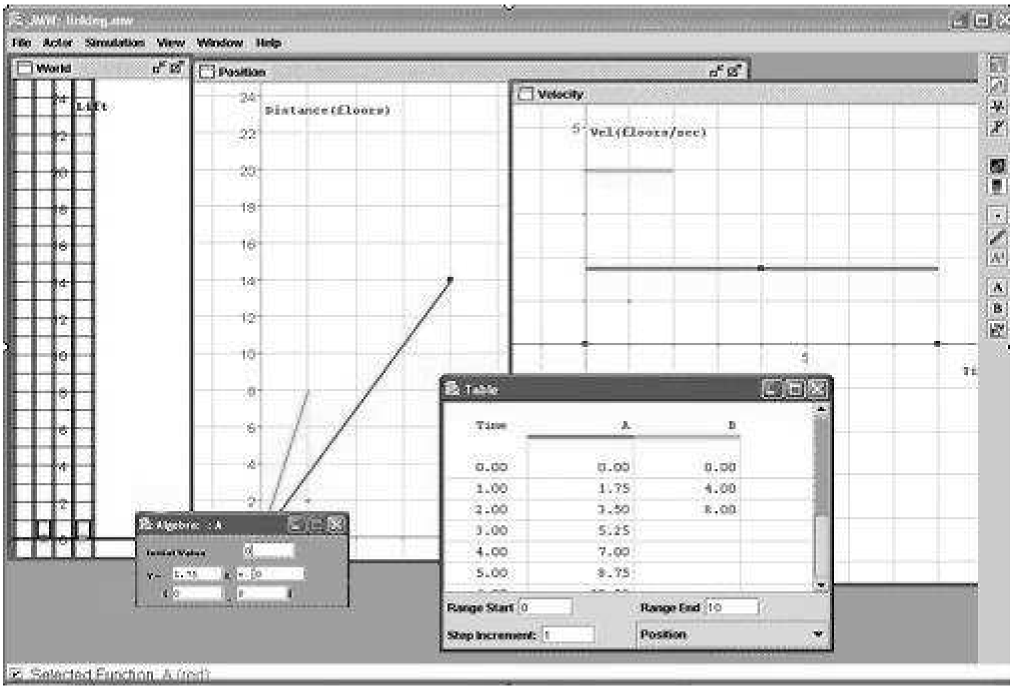

Figure 1 Screen dump of JavaMathWorlds showing multiple representations

The pre and post-tests considered rate of change in two motion contexts and one non-motion context Both vertical and horizontal motion contexts were included in the tests In the pre-test, the vertical motion scenario involved a tethered hot air balloon rising and falling It was intended that this scenario would parallel the vertical motion of the lift in MathWorlds Items related to this scenario probed interpretation of a position-time graph representing the movement of the balloon, average velocity, and interpretation of both a simple and a more complex velocity-time graph The horizontal motion scenario was the experientially real situation of a student walking to school and was included in order to establish the informal understandings of velocity brought by the students to the classroom As in the vertical context, the items involved position-time graphs, velocity-time graphs and average velocity However these items were more searching requiring description of the motion shown on a multi-segment position-time graph, drawing of graphs from a written description and an interpretation of a velocity-time graph representing two walkers moving at changing velocities Finally, the non-motion context described was the fee schedule for a computer technician Items related to this scenario were similar to those asked in the motion contexts with the addition of an item involving the symbolic representation

The post-test paralleled the pre-test but using different contexts and order of questions The contexts described were the platform of a window washer going up and down a tall building; a family on a long car trip during the school holidays; and the total cost of a vacuum cleaner with additional disposable dust-bags (Bardini, Pierce & Stacey, 2005) During data analysis like questions were paired for comparison

The next section presents an analysis based on students' post-test scripts and some illuminating student comments A discussion of some pertinent items will accompany the results

Results and Discussion

In the pre-test each student interviewed gave either an incorrect response or no response to a question requiring students to work out the technician's hourly rate from a graph of total cost against time worked However in their interviews immediately following the post-test these students made comments clearly demonstrating rate-related reasoning For example,

Student A: plus 3 each time for the number of dollars for each bag
Student B: It goes up by 3 for every bag bought
Student C: … the graph will go up $10 for each hour that he's worked so say for 1 hour it would be up here at 30 – it jumped up to 30 in the first hour and the second hour it would go to 40

This suggests that these students thought in terms of the change in one variable per unit change in the other variable rather than calculating a gradient using a formula

When looking at the graph of the total cost of a vacuum cleaner with additional dust bags against the number of dustbags, comments suggesting a strong connection for students between rate of change in the motion context of MathWorlds and rates of change in other contexts were made For example,

Researcher: If you saw this graph in MathWorlds what would it mean?
Student A: It starts off really quickly and goes slower and slower
Student B: The lift goes faster first and then slows down
Student C: The lift started going fast then slower – started getting slower

It seems the cognitive residue of MathWorlds was a 'model for' rate of change that students could use to talk about rate of change in both motion and non-motion contexts

It is interesting to note that only one student used the term 'gradient' His use of the term inferred rate-related reasoning rather than the result of a formula calculation

Researcher: If you saw this algebraic expression [C = 3n+60]in the Algebra window of Mathworlds what would it mean?
Student B: It has to start at 60 at x=0 and gradient of 3
Researcher: What did that mean (gradient of 3) as far as the lift is concerned?
Student B: How many floors it goes up by each minute

No student performed explicit gradient formula calculations; instead they used rate-related reasoning to discuss the speed of the lift All students moved smoothly between the multiple representations of both motion and non-motion contexts The class teacher commented that these students had previously studied linear functions in a traditional manner So, these students could be expected to have developed an association between gradient, rate of change and the calculation rule but in this technology-enhanced learning environment they demonstrated contextually appropriate rate-related reasoning

Analysis of the pre and post-tests includes boxplots as shown in Figure 2 The pair of boxplots on the left shows a comparison of the students' percentage for the pre-test with the students' percentage for the post-test The pair of boxplots on the right shows a comparison of the students' percentage for the non-motion question on the pre-test with the students' percentage for the non-motion question on the post-test These show an improvement in the students' results overall, and, in particular, despite all the teaching being focussed on motion contexts, an improvement on the students' percentages for the non-motion question Interestingly the pair of boxplots on the right shows, not only a marked improvement in students' scores on the non-motion questions, but also less variation in scores across the class

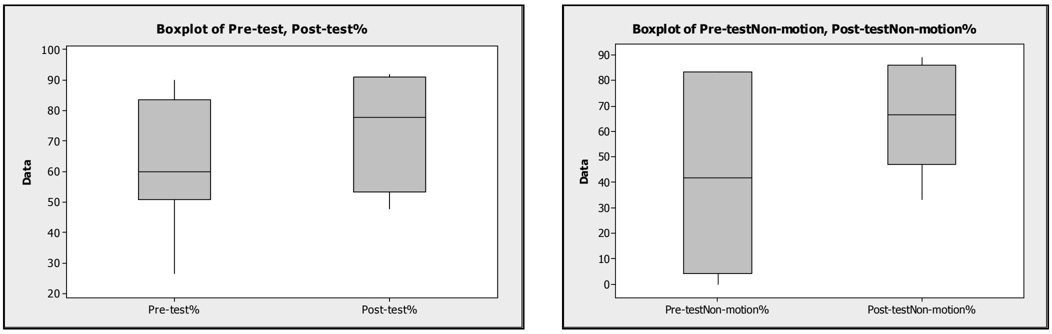

Figure 2 Boxplots illustrating improvement overall and in non-motion items

In the pre-test, students were asked to work out the technician's hourly rate from a graph of total cost against time worked Only one student was able to provide a correct response In a similar item from the post-test, students were asked to work out the cost per bag from a graph of total cost against number of bags Student responses showed a marked improvement in facility with this item with 75% able to give a correct response

In the pre-test students were given a symbolic expression representing the total cost of a computer technician's visit and asked for the hourly rate Half the students gave correct responses to this item In a similar item from the post-test, students were asked to work out the cost per bag from a symbolic expression representing the total cost Student responses again showed a marked improvement in facility with 83% able to give a correct response

Analysis of the pre and post-tests results suggests that these instruments could be shortened by removing the vertical motion scenario since using two motion contexts did not add to the data A closer parallel of the items in each scenario especially with regard to the numeric, graphical and symbolic representations would enable more detailed direct comparison of items The adaptation of the SimCalc project materials will also be further refined to allow students to work more independently

Conclusion

Analysis of the pre and post-tests results , illustrated in Figure 2, indicates that, even though the lessons did not include non-motion contexts, nevertheless, the students demonstrated an improved understanding of the concept of rate of change in both motion and non-motion contexts The 'models of' the motion of the lift developed to answer questions about the movement of a lift in MathWorlds appears to act as a 'model for' rate of change in the non-motion context This suggests that the use of the motion context of lifts in MathWorlds to develop numerous 'models of' rate of change has facilitated the development of an emergent model for rate of change in other contexts In this way velocity has become a 'model for' rate of change to solve problems in non-motion contexts thus expanding students' concept image of rate of change

This study supports the use of a technology enhanced learning environment to build on students' prior knowledge by providing an animated, visual and symbolic connection to an experientially real context The results of the post-test and interviews suggest that the cognitive residue of the learning was an expanded, correct concept image of rate of change Students clearly demonstrated the use of rate-related reasoning rather than formula calculations for gradient These encouraging results indicate that further research into the use of technology to develop rate-related reasoning is warranted

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