Years 2 and 3 children's strategies for mental addition and subtraction

T.J. Cooper, A. Heirdsfield and C.J. Irons

Centre for Mathematics and Science Education,

QUT, Kelvin Grove, Australia.

This paper reports on the correct responses and correct-response strategies for word problems from a longitudinal study of grades 2 and 3 children's mental computation strategies for 2 and 3 digit addition and subtraction. Children were found to use increasingly powerful strategies across the two years. However, the pen and paper algorithm tended to become dominant.

Trafton (1978, p.207) referred to mental arithmetic as "non-standard algorithms for computing exact answers" without the use of pen and paper. Sowder (1988, p.182) defined mental computation to be "the process of carrying out arithmetic calculations without the aid of external devices". She argued that mental procedures are often different from pen and paper algorithms, are not usually performed with pen and paper and have their numbers usually visible (Sowder, 1990). This paper adopts Sowder's definition of mental arithmetic, but, similar to Heirdsfield and Cooper (1995), reserves the term for mental calculations larger than those covered by basic facts, and allows the mental use of the traditional pen and paper algorithm, number-fact strategies and counting to be included.

A variety of mental strategies has been identified in the literature for addition and subtraction (Beishuizen, 1993; Carraher, Carraher, & Schliemann, 1987; Ginsberg, Posner, & Russell, 1981; Hope, 1987; Madell, 1985; Resnick, 1986) - categorisation based on Heirdsfield and Cooper (1995):

STRATEGIES		EXAMPLES
Simple	*Count on/back*	26+38: 26, 27, 28, ...
		53-24: 53, 52, 51, ...
Pen & paper	*u-1010*	26+38: 6+8=14; 20+30+10=60; 64
		53-24: 13-4=9; 40-20=20; 29
Left to right	*1010*	26+38: 20+30=50; 6+8=14; 64
		53-24: 50-20=30; 3-4='down' 1; 29
	Regrouping	26+38: (20+6)+(30+8)=(20+30)+(6+8)=50+(6+4)+4=64
Aggregation	*N10*	26+38: 26+30=56; 56+8=64
		53-24: 53-20=33- 33-4=29
	u-N10	26+8=34; 34+30=64
		53-24: 53-4=49; 49-20=29
Wholistic	*Decomposition*	272+37:272+28=300; 28+9=37; 300+9=309
		243-75: 75+25=100; 100+143=243; 143+25=168
	Compensation	56+98: =54+100=154
		334-189:=345-200=145

The sources for these strategies have been varied. Ginsberg, Posner, and Russel (1981) studied mental addition strategies for number computations (computations in symbolic form) presented verbally, of schooled and 'unschooled' American and Dioula (Ivory Coast) children and adults. The American subjects were 8 boys and 8 girls in each of grade 2 (mean age 8.1 years) and grade 5 (mean age 11.2), 60% white and 40% black from working and middle class backgrounds, from a public school in Ithaca, New York, and 16 college students from the University of Maryland (8 men and 8 women). The schooled Dioula were 8 boys and 8 girls in each of grade 3 (mean age 9.8) and grade 6 (mean age 12.6) and 12 adults from a private secondary evening school and 4 from a teacher training college. The unschooled Dioula were 8 boys and 8 girls aged 9-10 years and 8 boys and 8 girls aged 12-13 years and 16 adult merchants. Ginsberg et al. found that there was a high incidence of error for all children who used the pen and paper algorithm mentally. They also found that the 'unschooled' subjects tended to use the *regrouping* strategy, showing a good 'number sense'.

Madell (1985) investigated grade K to 3 children's addition and subtraction strategies for number computations at the Village Community School, Greenwich Village, where computation strategies are not taught until the end of grade 3. He documented the presence of the left to right and aggregation strategies. Resnick (1986) identified the presence of the *1010* and *compensation* strategies for number computations in a case study of a child aged 7 years. Hope (1987) identified the presence of the left to right strategies for number computations in a case study of a student aged 13 years who was a highly skilled mental computer. He argued that left to right strategies were memory-efficient because a series of discrete calculations does not have to be retained in short term memory and the order of the series does not have to be transformed back into a left to right sequence. He contended that it was easier to mentally combine than remember partial results, and, hence, continually updating calculations was efficient. Carraher et al. (1987) studied 16 Brazilian third graders' (aged 8 to 13 years) written and oral mathematics for the four operations using simulated store problems, story problems and number computations (the children had been taught the pen and paper algorithm). They observed that the children tended to use mental strategies, mainly left to right but including the *decomposition* strategy, for the simulated store and word problems, and pen and paper methods for the computation exercises. They also found that children who worked left to right produced more sensible answers. This finding was supported by Kamii (1989) who studied grade 2 children who had been encouraged to invent their own computation strategies. She found that these children tended to add the tens first. She further observed that, when children added the tens first, they made less errors and demonstrated more number sense when presented with computation with misaligned digits or in a horizontal form.

Beishuizen (1993) conducted a field study involving 125 high and low achieving middle-class grade 2 children in Holland (where mental strategies are taught) to investigate the effectiveness of the *1010, N10, u-N10* and *u-1010* (the traditional pen and paper algorithm) strategies for number computations. Two-digit addition and subtraction questions with and without regrouping were presented on cards. Beishuizen found that the *1010* strategy was favoured by low achievers but that the *N10* and *u-N10* strategies were more powerful in terms of supporting correct mental computation for larger and more complex numbers. This tendency for low achievers to use less powerful strategies was also reported by Hope (1985) who found that poor mental computers relied almost exclusively on the pen and paper algorithm.

In 1991, 1992 and 1993, an ARC funded longitudinal study of grades 2 and 3 children's mental computation strategies for addition and subtraction was undertaken by the authors. One-hundred and six children were presented with story problems and number computations (in horizontal and vertical form) for 2 and 3 digit addition and subtraction in six interviews from the beginning of grade 2 to the beginning of grade 4. This paper reports on the correct responses and correct-response strategies for the word problems.

Grades 2 and 3 were chosen because these are the years in which Queensland schools teach the pen and paper algorithm. It was hoped that effects of this instruction on spontaneous strategies would be observed.

Method

Subjects

The subjects were 106 children of varying mathematical abilities (one-third each of above average, average and below average) in 6 primary schools (3 state and 3 Catholic) representing a variety of social backgrounds. They participated in the study for two years (from the beginning of grade 2 to the beginning of grade 4).

Instruments

The instrument used was Piaget's revised clinical interview technique (Ginsburg, Kossan, Schwartz, & Swanson, 1983). Newell and Simon's (1972) talk aloud technique was tried but found unsuccessful. The interview questions covered in this paper consisted of a series of 2 and 3 digit addition and subtraction word problems, presented in visual and oral form (the interviewer read the questions). The word problems covered the following types; (1) joining addition, "If the apple cost 35 cents and the orange 27 cents, how much did both cost?"; (2) separation subtraction, "If John had 82 cents and spent 54 cents on bananas, how much money does he have left?"; and (3) missing-addend, "If Nancy had 47 cents and the chocolate costs 75 cents, how much extra money does she need?". The question types were as follows:

- A1: addition 2 digits no regrouping
- A2: addition 2 digits regrouping
- A3: addition 3 digits no regrouping
- A4: addition 3 digits regrouping
- S1: separation subtraction 2 digits no regrouping
- S2: separation subtraction 2 digits regrouping
- S3: separation subtraction 3 digits no regrouping
- S4: separation subtraction 3 digits regrouping
- M1: missing-addend subtraction 2 digits no regrouping
- M2: missing-addend subtraction 2 digits regrouping

The structure of the interview was to give the questions for each problem type in increasing order of difficulty until the children's responses indicated they were experiencing difficulties. The previous responses of the children (in the given or previous interview) assisted the interviewer find appropriate starting levels.

Procedure

The children were interviewed 6 times: the beginning, middle and end of grade 2 (interviews 1, 2 and 3), the beginning and end of grade 3 (interviews 4 and 5), and the beginning of grade 4 (interview 6). The children were withdrawn from the classroom and interviewed in a separate room. The interviews were videotaped. The length of the interviews was kept to approximately 30 minutes.

The word problems were revised after interview 3 to take account of children's knowledge growth, to allow the interviewer to probe for the existence of the highest level strategies and to keep the interview length to 30 minutes. The interviewer also attempted to ask questions at the level of each child. This meant that after interview 4, the better students were not asked the simpler question types and that the simpler question types contained only the more difficult examples from this type.

Results

The videotapes were transcribed into protocols and behaviours analysed for strategy categories. After negotiation among three researchers, the following categories were identified:

CODE	DESCRIPTION OF STRATEGY CATEGORY
1	*counting* (counting both numbers by modelling with fingers; counting on/back with modelling; counting on/back without modelling), e.g., 27+15: 27, 28, 29, 30, ...
2	*number fact strategies* (e.g., near doubles; near 10; derived from known fact; known fact), e.g., 15+17: 15+15+2=30+2=32
3	*u-1010* (pen and paper algorithm where numbers are separated into place values; procedure right to left), e.g., 28+35: 5+8=13=10+3; 20+30+10=60; 63
4	*1010* (numbers separated into place values; procedure left to right), e.g., 28+35: 20+30=50; 5+8=13; 50+13=63
5	*u-N10* (one number kept whole; other number separated into place values; procedure adding right to left - aggregated, separated and R-L aggregated), e.g., 28+35: 28+5=33; 33+30=63
6	*N10* (one number kept whole; other number separated into place values; procedure adding left to right - aggregated, separated and L-R aggregated), e.g., 28+35: 28+30=58; 58+5=63
7	*mixed* (a mixture of aggregation and separated place values involving left to right and right to left, also where students employed a variety of strategies for similar question types), e.g., 368+275: 368+200=568; 568+5=573; 573+60=633
8	*wholistic* (decomposition and regrouping, and mixtures with earlier strategies), e.g., 38+56: 40+50+4=94

The interview questions were coded using the above categories and these coded responses tabulated and analysed. Analysis was aimed at identifying changes in use of strategies over the two years.

The tables below show children's responses for the six interviews. It should be noted that, for all interviews except 6, some question types were missed; and that, for interview 3, 26 children were not interviewed. Low percentages for simpler question types in interviews 5 and 6 should be seen in terms of these questions only being asked of the poorer performing children.

INTERVIEW 1 (106 children) Question type

		A1	A2	A3	A4	S1	S2	S3	S4	M1	M2
%attempted		54.7	9.4			41.5	5.7			34.0	1.9
%correct		35.8	7.5			16.0	2.8			14.2	1.0
%using each	1	25.5	0			8.5	0			9.5	0
strategy	2	0	0			0	0			0.9	0
correctly	3	0	0.9			0	0.9			2.8	0
	4	5.4	1.9			2.8	1.9			0	0
	5	0	0			0	0			0	0
	6	1.9	3.8			2.8	0			1.0	0
	7	1.9	0			1.9	0			0	0
	8	0.9	0.9			0	0			0	1.0

INTERVIEW 2 (106children) Question type

		A1	A2	A3	A4	S1	S2	S3	S4	M1	M2
%attempted		72.6	27.4			61.3	14.2			50.0	3.8
%correct		54.7	14.2			32.1	3.8			30.2	1.0
%using each	1	19.8	0			5.6	0			8.5	0
strategy	2	0	0			0	0			0.9	0
correctly	3	15.1	3.8			8.5	0			1.9	0
	4	16.0	7.6			14.2	1.0			16.0	0
	5	0.9	0			0	0			0	0
	6	1.9	1.9			1.9	3.0			0.9	0
	7	0.9	1.0			0.9	0			1.9	1.0
	8	0	0			0.9	0			0	0

INTERVIEW 3 (90 children) Question type

		A1	A2	A3	A4	S1	S2	S3	S4	M1	M2
%attempted		77.8	32.2			73.3	21.1			55.6	5.6
%correct		57.8	22.2			41.1	6.7			31.1	0
%using each	1	16.6	2.2			7.8	1.1			4.4	0
strategy	2	0	0			0	0			0	0
correctly	3	11.1	6.7			6.7	1.1			8.9	0
	4	23.3	5.6			15.5	1.1			14.4	0
	5	0	0			0	0			0	0
	6	5.5	6.7			1.1	2.2			1.1	0
	7	1.1	1.1			8.9	1.1			0	0
	8	0	0			1.1	0			0	0

INTERVIEW 4 (106 children) Question type

		A1	A2	A3	A4	S1	S2	S3	S4	M1	M2
%attempted		94.3	69.8		18.9	79.2	50.9		2.8		
%correct		85.8	56.6		14.2	65.1	17.9		2.8		
%usingeach	1	9.4	1.0		0	1.9	0		0		
strategy	2	2.8	0		0	3.8	0		0		
correctly	3	25.5	17.9		0	21.7	1.9		0		
	4	44.3	17.9		0.06	34.0	5.7		0.9		
	5	0	0		0	0	0		0		
	6	0	5.7		0.06	1.9	5.7		0		
	7	2.8	12.3		0.2	1.9	3.8		0		
	8	0.9	1.0		0.6	0	0.9		1.9		

INTERVIEW 5 (106 children) Question type

		A1	A2	A3	A4	S1	S2	S3	S4	M1	M2
%attempted		37.7	84.0		34.9	71.7	69.8		15.1	57.5	20.8
%correct		32.1	59.4		23.6	54.7	20.8		10.4	47.2	10.4
%using each	1	0	0		0	4.7	0.9		0	3.8	0.9
strategy	2	1.0	0		0	2.8	0		0	2.8	0
correctly	3	14.1	27.3		6.6	11.0	8.5		0.9	7.6	0.9
	4	16.1	20.7		3.8	25.5	3.7		0.9	26.4	0
	5	0	0		0	0	0		0.9	0	2.8
	6	0	1.9		1.9	0	4.7		2.8	4.7	0.9
	7	1.0	8.5		1.9	5.6	2.8		0.9	0	2.8
	8	0	1.0		9.4	4.7	0		3.8	1.9	1.9

INTERVIEW6(106children) Questiontype

		A1	A2	A3	A4	S1	S2	S3	S4	M1	M2
%attempted		21.7	91.5	82.1	68.9	64.2	78.3	72.6	28.3	42.5	37.8
%correct		17.9	71.7	75.5	42.5	53.8	40.6	64.2	14.2	30.2	22.6
%using each	1	2.0	0.9	0	0	1.9	0	0	0	0	0.9
strategy	2	0	0	0	0	0	0	0	0	0	0
correctly	3	7.5	40.6	44.4	30.2	18.9	21.7	34.0	5.7	8.5	9.4
	4	7.5	14.1	28.3	3.8	25.5	3.8	11.3	1.9	17.9	3.8
	5	0	0	0	0	0	0	0	0	0	1.9
	6	0	1.9	0	0	1.9	3.8	1.0	1.0	3.8	2.8
	7	0.9	14.1	2.9	1.9	3.8	5.7	14.1	1.9	0	1.9
	8	0	0	0	6.6	1.9	5.7	3.8	3.8	0	1.9

Discussion and conclusions

In the literature, only Carraher et al. (1987) used word problems in studies of children's mental computation strategies. All other researchers used number computations (i.e. symbols only). In a study of mental computation performance in grades 2, 5 and 7, Reys, Reys, and Hope (1993) presented applied problems, but they collected written answers and could not identify the strategy or the cause of error. Hence, this present study appears to break new ground.

The information in the tables indicates that children used a variety of strategies and that these strategies changed across the interviews. Moreover, the percentage of children attempting more difficult questions and the percentage of children who were correct in more difficult questions increased over the interviews. The percentage of children correctly using strategy 1 (*count*) decreased over the interviews (although it was still evident in interview 6 for the simpler examples which were being attempted by the poorer students), while the percentage of children correctly using more powerful strategies increased. In particular, strategies 4 and 8 (*1010* and*wholistic*) increased their popularity. The aggregation strategies, 5 and 6, were less popular than the separated strategies 3 and 4; however, strategy 6 (*N10*) was used more than strategy 5 (*u-N10*). The preference for separated strategies (3 and 4) over aggregation (5 and 6) was also reported in the study by Beishuizen (1993).

However, the most interesting change was in strategies 3 and 4 (*u-1010* and *1010*). Initially, *1010* (left to right) replaced count as the most popular correctly-used strategy, although by the second interview, *u-1010* (right to left) was also evident. Whereas, by interview 6, *u-1010* was the dominant correctly-used strategy. This is particularly so for the regrouping examples, which appears to indicate an instructional effect. Throughout the interviews, some students switched between left to right and right to left (strategy 7) in the same example types and within the one example.

With regard to question types, attempts and correct answers to missing-addend subtraction question types (M1 and M2) were significantly lower than addition and separation subtraction question types (A1-A4 and S1-S4), indicating the difficulty that children have with this meaning of subtraction. Further, children were less successful with subtraction than addition, suggesting difficulties with the concept of subtraction in general.

References

Beishuizen, M. (1993). Mental strategies and materials or models for addition and subtraction up to 100 in Dutch second grades. Journal for Research in Mathematics Education, 24(4), *294-323.*

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1987). Written and oral mathematics. Journal for Research in Mathematics Education, 18(2), *83-87.*

Ginsburg, H. P., Posner, J. K., & Russell, R. L. (1981). The development of mental addition as a function of schooling and culture. Journal of Cross Cultural Psychology, 12(2), *163-178.*

Heirdsfield, A.M., & Cooper, T.J. (1995, July). Teaching implications from research on children's mental computation for addition and subtraction. Proceedings of the 15th National Conference of the Australian Association of Mathematics Teachers. *Darwin, NT.*

Hope, J. A. (1985). Unravelling the mysteries of expert mental calculation. Educational Studies in Mathematics, 16, *355-374.*

Hope, J. A. (1987). A case of a highly skilled mental calculator. Journal for Research in Mathematics Education, 18(5), *331-342.*

Kamii, C. (1989). Young children continue to reinvent arithmetic - 2nd grade: implications of Piaget's theory. *New York. Teachers College Press.*

Ginsburg, H. P., Kossan, N. E., Schwartz, R., & Swanson, D. (1983). Protocol methods in research on mathematical thinking. In H. P. Ginsburg (Ed.), The development of mathematical thinking. *New York: Academic.*

Madell, R. (1985). Children's natural processes. Arithmetic Teacher, 32(7), *20-22.*

Newell, A., & Simon, H. A. (1972). Human problem solving. *Engelwood Cliffs, New Jersey: Prentice Hall.*

Resnick, L. B. (1986). The development of mathematical intuition. In M. Perlmutter (Ed.), Perspectives on intellectual development. The Minnesota Symposia on Child Psychology. Volume 19. *Hillsdale, New Jersey: Lawrence Erlbaum Associates, Publishers.*

Reys, B. J., Reys, R. E., & Hope, J. A. (1993). Mental computation: A snapshot of 2nd, 5th, and 7th grade student performance. School Science and Mathematics. 93(6). 306-315.

Sowder, J. (1988). Making sense of numbers in school mathematics. In G. Leinhardt, R. Putman, & R. Hattrup (Eds.), Analysis of arithmetic for mathematics. *Hillsdale, N J: Erlbaum.*

Sowder, J. (1990). Mental computation and number sense. Arithmetic Teacher, 37(7), *18-20.*

Trafton, P. (1978). Estimation and mental computation: Important components of computation. In 1978 NCTM Yearbook. *Reston: NCTM.*