ALGEBRA: BEYOND MANIPULATING SYMBOLS

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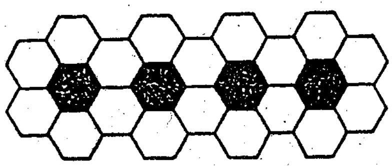
Over the last two decades a vast number of research projects have identified areas of students' misunderstandings in the algebraic domain. It appears that the main focus of this research has been on developing an understanding of variables, the translation of word problems, and on "doing" algebra - the manipulation of symbols. Although these aspects are important to algebra, Booth (1989) and Kieran (1989) believe that a critical aspect is understanding just what the algebraic statement represents, both visually and symbolically. When linking and using various representations, spatial skills and higher order thinking skills play key roles, and are therefore crucial to the acquisition of algebraic understanding. This paper illustrates the importance of these skills to the algebraic domain, reviews the literature pertaining to these skills within the algebraic domain, and identifies the research implications drawn from this literature.

RESEARCH DONE ON ALGEBRA

Research in the algebraic domain has identified many areas of students' misconceptions. Children appear to have difficulty with the variable concept (Booth, 1988; Kuchemann, 1981; Usiskin, 1988), the visual syntax of algebra (Bennett, 1988; Chalouh and Herscovics, 1988; Kirshner, 1989), the concatenation of algebraic expressions (Chalouh and Herscovics, 1988), the changing nature of the equal sign (Davis, 1989 & Kieran, 1989), and the manipulation of symbols (Wheeler, 1989). This research reflects the most prevalent approach historically used for introducing algebra to the beginning student, linking algebra to arithmetic. For this approach the focus seems to be on the acquisition of routine skills and procedures or algorithmic methods (Wheeler 1989), with a heavy reliance on the manipulation of symbols (Davis, 1989). Thus the focus of past research has been narrow and limited with the majority of projects being mainly concerned with the algebraic symbol and its manipulation.

A more recent approach for teaching algebra uses patterning from which algebraic expressions are generated (Bennet 1988). This approach entails introducing algebra by looking at patterns, creating tables, describing the pattern, and "short handing" these descriptions into algebra. The following example typifies this approach.



The council with to create flower beds and surround them with hexagonal paving slabs according to the pattern shown above. Find a formula that the council can use to decide the number of slabs needed for any number of flower beds.

In order to reach an appropriate solution students are required to proceed through a number of steps. They need to see the pattern in terms of its component parts (one flower bed, two flower beds etc.), identify the generalities within these parts (e.g. each new flower bed requires four extra tiles), and extend the pattern using these generalities (for every flower bed there are four tiles plus the two at the beginning). These generalities then need to be transformed into a suitable formula. Success for this problem appears to involve more than an ability to manipulate symbols. Success not only requires sound visual skills (seeing the pattern, dissecting the pattern into its component parts) but also an array of specific thinking skills (mentally rotating the pattern, logical reasoning, an ability to complete the pattern etc.).

Ursini (1991) found that this approach is fraught with difficulties. He reported that, in their attempts to generalise, children did not link the two proposed representations, could not express a generalisation, and disregarded all patterns when trying to generalise. Even though the approach attempts to link algebraic symbols to visual patterns students tend to use procedural means for reaching a solution e.g. "Well we were taught that you draw a table and work it out...you use trial and error...first powers, then square numbers, then triangle numbers...". Stacey (1988) found most students, when looking for patterns from tables tended to use the strategy of guess and check, concentrating on repeated addition, multiplication and simple ratio relationships. The reflex of checking the formula against the given (usually visual) data is not present (Lee and Wheeler, 1989). Yet Booth (1989) and Kieran (1989) believe that a critical aspect in algebra is understanding just what the algebraic statement represents, both visually and symbolically. Endeavouring to link visual and symbolic representations is important and yet it seems that the majority of students are failing to achieve this goal. The role of thinking skills and spatial skills in attaining this aim needs to be explored. Some questions that need to be addressed are:-

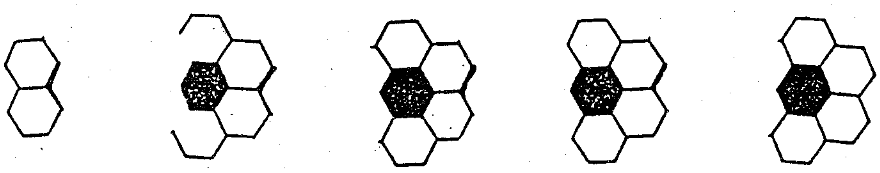
How important is it for students to make these links in the development of algebraic understanding?

Why are students failing to make these links?

What role do visual skills play in forging these connections?

What thinking skills enhance the formation of these links?

The importance of the establishment of these links can be drawn from the literature pertaining to expert vs novice algebraists. Blais (1988) believes the major difference between the novice and expert algebraist is that the expert perceives "essence". In the case of algebraic simplification the expert perceives visual forms of the representation along with informal English. Blais (1988, p 626) states that "since novices are usually symbolically illiterate, communication restricted to the use of formal symbolic expressions has the effect of concealing essence from them." Lesh, Post, and Behr (1987) believe that in order to understand an idea one must first recognise the idea embedded in a variety of representational systems, flexibly manipulate the idea within these systems, and accurately translate the idea from one system to another. Good problem solvers tend to be sufficiently flexible in their use of a variety of relevant representation systems, and instinctively switch to the most convenient representation to emphasise the solution process (Jones, Palinscar, Ogle, & Carr, 1987). However, Lesh et al (1987) maintain that students rarely seem to get things into a single coherent representation of the problem so that they can process the information. Thus is it not only important to develop a variety of representations for a problem but also to understand how these representations interact with each other. Students not only need to attain algebraic expressions from visual patterns but also to interpret these expressions in terms of the visual data. For the "flower bed" problem students need to see the following pattern representing the expression 2 + 4n.



Further they need to be able to relate expressions such as 2(n+1) + 2n to the pattern. Thus the role visual skills play in forging these connections needs to be explored.

In the literature, visualisation appears to consist of two distinct interpretations. Zimmermann and Cunningham (1991) define visualisation as the process of producing and using geometrical or graphical representations of mathematical concepts, principles and problems. This contrasts with the psychological studies which tend to focus on the subject's ability to form and manipulate mental images. The literature on visualisation identifies students with differing visual abilities. Krutetskii (1976) claims that individuals can be classified into three main groups with respect to visual ability. These are "geometric type, analytic type,and harmonic type" (Krutetskii, 1976) or more loosely termed "visualisers, verbalisers and mixers" (Clements, 1982). Individuals are categorised according to the processes they use in problem solving. Visualisers habitually employ imagery or pictorial notations, verbalisers tend to use verbal codes rather than visual images or pictorial notations, and mixers tend to use a mixture of both linguistic and spatial strategies. The preponderance of students appear to be verbalisers (Clements, 1982).

Ben-Haim (1983), and Fennema and Sherman (1977) found visual ability shows about the same magnitude of correlation with mathematical achievement as the verbal component. By contrast, Krutetskii (1976), Presmeg (1986), and Lean and Clements (1981) conclude that there is a tendency for students who prefer to process mathematical information by verbal logical means to outperform more visual students on both mathematical and spatial tests. They felt that these findings reflect the types of mathematics presented in the majority of classrooms, and in fact students may be highly successful in learning school mathematics without needing to resort to visual thinking. Two main reasons were proffered to explain this phenomena. Firstly, in most curriculum, thinking visually is not deemed important. Presmeg (1986) maintains that teachers and the curriculum often present visual reasoning as an introduction, auxiliary argument, or an accessory. Secondly, visual methods are time consuming and the time constraints of testing procedures militate against extensive use of visual methods (Clement, 1982; Eisenberg & Dreyfus, 1991; Presmeg, 1986). Hence the types of mathematics being taught and tested in schools appears to favour the verbalisers and therefore it is not surprising to find many of these being high achievers under the present curriculum constraints.

Presmeg (1986) reports that when a topic is first taught, a visual presentation aids visualisers' understanding but practice of the procedure or formula leads to habituation rendering the use of imagery unnecessary. Eisenberg and Dreyfus (1986) support this finding and suggest that for many students visual thinking and analytical thinking seem to be dichotomous modes, with the analytical mode being overwhelmingly stronger. Thus, for these students, being able to visualise may not be a crucial requisite for success. Eisenberg and Dreyfus (1991) claim thinking visually makes higher cognitive demands than thinking algorithmically and thus wherever possible students tend to choose a symbolic framework for processing mathematical information. They found that even if students were forced to visually process they still preferred to think symbolically. Lean and Clements' (1981) research supports this finding. They maintain that in some situations visualisation can have a detrimental effect on abstract conceptualisation as visualisation can cause memory "overload" and result in slow retrieval during recall. Though visualisation is considered to be a difficult process, it plays a key role when linking and using various representation in problem solving (Schonberger, 1981). Even though this role has been acknowledged the amount of research carried out in this area has been minimal. Of particular interest to this research is the role it plays in pattern generalisation and interpretation, and more specifically how children with differing spatial abilities approach and solve pattern generating problems.

Any transformation of information is considered to involve higher-order thinking skills. The literature reflects difficulties encountered in defining the term, "thinking". Many researchers describe thinking in terms of the features it encompasses. Resnick (1987) identifies some of the key features of higher order thinking as nonalgorithmic and complex, involving interpretation, uncertainty, applying multiple criteria, and imposing meaning. It is effortful and depends on self-regulation. Linking problem representations seems to invoke particular thinking skills. The literature identifies some specific thinking skills that enhance this process. It seems that developing flexibility in mode of representation involves developing an array of mathematical thinking processes. Some of the key processes crucial to flexibility are thought to be spatial thinking including a facility with mental rotation, logical and analogical reasoning, classifying and hypothesising, and an ability to complete pattern and generalise (Lipman, 1985). Thus the interaction between these specific thinking skills and visual ability, and the role they play in enhancing the formation of links between various representations, needs to be explored. This research project endeavours to address some of these issues. Of particular interest is how visual ability and thinking skills, such as logical reasoning, analogical reasoning, pattern completion and mental rotation, enhance and support the elaboration of simple algebraic expressions, and whether differing visual abilities reflect differing capabilities in this linking and elaborating process. Some of the specific aims of the research are as follows:-

1. to identify students' dominant mode or thinking i.e. visual, verbal or mixers.
2. to investigate any relationships between students' dominant thinking mode and transfer between knowledge representations in linear algebraic situations i.e.
(a) do visual thinkers prefer transfer from visual representation to symbolic representation?
(b) do symbolic thinkers prefer transfer from symbolic representation to visual representation?
3. to identify possible relationships between student's general thinking skills and their competence in transferring from one form of algebraic representation to the other.
4. to investigate whether these higher-order thinking skills differ for the two dominant thinking modes (visual and symbolic).

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