A Mathematics Education Ghost Story: Herbartianism and School Mathematics

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In the 1890s "Herbartianism", a modified form of the education theory of Johann Friedrich Herbart, became the dominant influence on elementary school education on the Continent, in Great Britain, and in the United States. Herbartianism burst on the education scene, meteorlike, and then, after about 20 years, virtually disappeared. In this paper its influence on school mathematics is assessed. It is argued that within Herbartianism lay the seeds of most of the 20th-century's major mathematics education theories.

In the course of commenting on the educational theories of the German philosopher and educationist Johann Friedrich Herbart (1776–1841), Richard Selleck (1968) stated that whatever reservations commentators might have of Herbart's views, "his work has a complexity, subtlety and coherence which make it more impressive than the writings of comparative amateurs such as Froebel or Pestalozzi" (p. 227). Such an assessment is hardly an exaggeration, for Herbart was a philosopher good enough to hold the Chair in Philosophy at Königsberg University soon after it had been held by Immanuel Kant.

During the period 1850–1910 there arose groups of scholars who propagated their versions of what Herbart had said about education. The first set of scholars were in Herbart's homeland, Germany, and the most notable among these were Karl Vomar Stoy, at the University of Jena, Tuiskon Ziller at a pedagogical seminary in Leipzig, and William Rein at the University of Jena. These scholars modified and interpreted the philosophy and educational theory of Herbart. In their turn, they greatly influenced educators from abroad who came to study in Germany. These visitors included John Adams (1897) and Frank Hayward (1904) from Great Britain, Charles De Garmo (1889, 1895), Charles McMurry (1895, 1906) and his brother, Frank McMurry (McMurry & McMurry, 1897) from the United States, and John Smyth from Australia (Selleck, 1968). Percival Cole (1907, 1912), an Australian, came under the influence of Herbartianism during periods of study in London and New York, and brought back his own brand of Herbartian zeal to Sydney Teachers Training College where, for many years, he served as Vice-Principal.

Our title for this paper was inspired by the title of Harold B. Dunkel's (1970) scholarly book, *Herbart and Herbartianism: An Educational Ghost Story*, which traced the influence of Herbartianism in the United States. Dunkel, in seeking to explain why Herbartianism's "fame blazed up like a meteor and meteorlike was extinguished" (p. 4), stated that the purpose of his book was "to disentangle Herbart from the Herbartians and to see each element for what it actually was" (p. 16). He continued:

> If we honor Herbart merely as the alleged father of Herbartianism, then this procedure will, to a degree, lay the ghost of Herbart in the history of educational thought. But at least we will have materialized him as a philosopher and an educator into a more definite substance than that spectral figure to whom the incantations of the Herbartians are occasionally addressed. (pp. 16–17)

Dunkel wrote those words 35 years ago, but, subsequently, there have been few signs anywhere that teachers and educators have become more knowledgeable about Herbart's theory of education. In the year 2005, Herbart's education messages are unknown to most teachers. But, we shall argue, the ghosts of Herbart and Herbartianism are lurking in most

school classrooms, including mathematics classrooms, guiding the subconscious planning and teaching behaviours of teachers, and contributing much to the so-called didactical contracts that teachers and their students subconsciously share.

We shall argue that not only was there much in the education writings of Herbart and the Herbartians that was ahead of its time, but in some ways they anticipated many key theoretical stances of mathematics educators working in the second half of the 20th century. Although these writers did not acknowledge a debt to Herbart or the Herbartians, that was probably because they were unaware of what Herbart had said and did not know that what they were saying had, to a certain extent, been said more than a century ago.

The Continuing Influence of Herbart's Ideas on Lesson Planning and Classroom Teaching

Although the Herbartians attempted to remain true to the principal ideas of the master, they consciously distilled them for an intended audience of practising, and prospective, teachers. The result, in the view of both Dunkel (1970) and Selleck (1968), was that publications of the Herbartians typically featured a mixture of simplified theory and advice for practice. This combination was well captured in the table of contents in Charles McMurry's (1895) *Elements of General Method Based on the Principles of Herbart*, which quickly became a best seller in the United States. The chapter titles listed were "The Aims of Education", "Relative Values of Studies", "Nature of Interest", "Concentration", "Induction", "Apperception", "The Will", and "Herbart and His Disciples" (p. 6).

The Formal Stages of Instruction

Central to Herbart's pedagogical thesis were his four steps of instruction: clarity, association, system, and method. According to Dunkel (1970), Herbart was not consistent in how he defined the four steps, or even in the terminology he used to describe them. But commentators (e.g., De Garmo, 1895; McMurry, 1895; Dunkel, 1970; Selleck, 1968) tend to agree that in his first step the teacher was called upon to distinguish sharply one concept from another, so that the student's attention would be focused on a single object (Herbart called this act of focusing, "concentration"). Then the student's thinking had to be expanded beyond the primary matter of attention and led, by the teacher, though the remaining stages (which, together, Herbart, referred to as the "reflection stages").

Ziller renamed Herbart's *four* steps Clearness, Association, System and Method, and Rein subsequently separated the clearness stage into two stages. Rein (1893) named the *five* steps (a) Preparation (analysis), (b) Presentation (synthesis), (c) Association, (d) Generalisation, and (e) Application. These can be summarised briefly in the following way (the summary is based on Rein's (1893) analysis of the essence of Herbart's steps):

1. *Preparation.* The willingness of an individual's mind to accept new ideas was totally dependent on the ways ideas already in that individual's mind were organized. Therefore, class instruction needed to aim at points of contact between the students, and to work at bringing those points of contact harmoniously together. Once that had been achieved a lesson that took account of the reasonably common introductory state of the students' minds could take place.
2. *Presentation.* New material that built on the old should be introduced in a logical manner, so that the planned concepts would most likely be "welcomed" into the students' minds. According to Herbart (1904a), if the presentation was a success,

"the reproduction by the pupils would show that they recall not merely the main facts but largely even the teacher's language" (pp. 108–109).

3. *Association.* Instruction is to be directed at facilitating the newly welcomed ideas (in a student's mind) so that they are associated with the student's whole circle of thinking. This will be achieved when the new ideas can be linked with "every point, forward, backwards, sideways" (Herbart, 1904a, pp. 55–56).
4. *Generalisation.* The teacher's intention should be to enable students to make leaps of generalisation so that they create more a general idea that had not been explicitly part of how the mind was previously organised.
5. *Application.* Instruction should be directed at encouraging students to apply the new idea, and the new mental organization, in contexts different from those in which they had been learned (Herbart, 1904b, p. 107).

As commentators (e.g., Dunkel, 1970; Selleck, 1968) have pointed out, Herbart and his followers could not agree on the names, or even on the definitions, of the steps. Nonetheless, the above description represents a reasonable account of how most Herbartians (e.g., De Garmo, 1895; McMurry, 1895) interpreted Herbart's steps. For teachers, the steps possessed an inherent plausibility, something that could, relatively easily, be put into practice when planning and presenting ordinary lessons (Dunkel, 1970; Selleck, 1968). When in 1900, David Eugene Smith (1900) was called upon to recommend texts that would be the most helpful for elementary school teachers wishing to read about teaching methods for mathematics lessons, he recommended only four books, and three of those were written by four members of the National Herbartian Society that existed at that time – the four were Charles De Garmo, Charles and Frank McMurry and John Dewey.

In the second half of the 1890s in Great Britain and in the United States the main concepts of Herbartianism became standard fare for those teacher educators in teachers colleges and normal schools who were responsible for teaching prospective teachers how to teach. The leaders of Herbartianism in the United States, Charles De Garmo and Charles and Frank McMurry, were all graduates of Illinois State Normal University (ISNU), and all three became Faculty members at ISNU. Each gained a doctorate in Germany working under William Rein, and each was destined to make a large contribution to the process of spreading the Herbartian gospel, as Rein interpreted it, in the United States. The young teachers whom they subsequently imbued with the spirit of Herbart and Rein may not have grasped the complexities of Herbartian doctrine, but they did learn to plan and deliver lessons according to the five steps (or at least as they perceived the five steps).

De Garmo's and Charles McMurrys' influence was not confined to ISNU, for their books on Herbartianism were runaway best sellers. Two years after its initial publication in 1893, Charles McMurray's (1895) *The Elements of General Method Based on the Principles of Herbart* was into its fifth edition, with 30 000 copies already having been sold. De Garmo's (1889), *The Essentials of Method* also sold over 30 000 copies (Dunkel, 1970). De Garmo would become Professor of Education at Illinois University, President of Swarthmore College, and Professor of Education at Cornell University, and both Charles and Frank McMurry would become Professors of Education at Columbia Teachers College in New York. All three would serve as key officers of the National Herbart Society. Their influence was pervasive, and in 1900 "normal colleges" all over the United States were training prospective teachers to plan and teach lessons according to the "five steps".

Selleck (1968) colourfully maintained that in England, "if the teachers of this period had been submitted to a projective test and given the cue, 'Herbart', there seems little doubt

that the most common response would have been, 'the five steps'" (p. 244). Selleck quoted an Inspector who wrote in his memoirs that every oral lesson conformed strictly to pattern and "the fashionable pattern at that time was that of the Herbartian Five Formal Steps". Another writer maintained that "all students of education were stepping to the tune of the Five Steps" (p. 245). Selleck (1968) went on to say:

> Not quite as quickly as they became popular, but with depressing speed nevertheless, the steps assumed a rigidity that was wholly undesirable. What had been intended to deliver the teacher from loose and ineffective procedures often resulted in another, though probably superior, form of slavery. Fennell's *Notes of Lessons on the Herbartian Method* provides an example of how sterile the steps had become. The book has almost 280 pages, of which 270 are devoted to specimen lessons built up on the model of the steps. ... [In the book] the steps are outlined incorrectly, the third and fourth steps being confused, and the fifth step being rendered as "Recapitulation". (p. 245)

The same kind of thing happened in Australia where Dr John Smyth (another former student of William Rein) became Principal of the Training College in Melbourne, and Percival R. Cole, a major contributor to the literature on Herbartianism (see, e.g., Cole, 1907, 1912) became Vice-Principal of the Training College in Sydney.

Although the influence of Herbartianism on the planning and practices of many young teachers was great, the context in which it occurred needs to be taken into account. In the 1890s and 1900s almost all of the formal preservice teacher education on the Continent, in Great Britain, in the United States, and in Australia took place in the teachers colleges (or "normal schools", as they were often called). As Herbartianism lost its steam during the first two decades of the 20th century, "Herbart's five steps" gradually became "the five steps", and a range of variants of the five steps came to be taught to new generations of teacher education students. That a German philosopher named Herbart had had something to do with planning steps for lessons became less and less known among teacher educators, including those who trained future mathematics teachers. Many different versions of the five steps were taught, and young teachers whose teaching was being evaluated during "teaching practice' rounds were expected to demonstrate an ability to apply the steps usefully. And so, the ghosts of Herbart and Herbartianism were able to exert a secret but extraordinarily powerful continuing influence on schools.

The Doctrine of Apperception and Mathematics Education

A mathematics educator in the year 2005 reading about the Herbartian concept of apperception is likely to be impressed by the way some of the important theoretical positions of constructivist-oriented 20th-century theorists – such as Piaget and Vygotsky – were expressed in the 19th century. John Adams (1897), of the Educational Institute of Scotland (and, later, Professor of Education at London University) gave the following definition of apperception as it was used in Herbartian psychology:

> Apperception means the acting upon a new idea by all the ideas present in the soul, and since the number and arrangement of ideas no two souls are exactly alike, it follows that no two persons can have precisely the same idea about anything. (p. 65)

Adams continued by stating that that almost every teacher thinks that when he shows a thing to his class, he has done the highest, the best, the ultimate, in teaching. He then quoted Jacotot, a French writer on the teaching of arithmetic:

> "What is a master?" he asks scornfully. "Isn't he a man who asks another – Don't you see what I am showing you?" Being in an oratorical mood, Jacotot does not pause for a reply. The schoolmaster in his work is not in such a hurry, and insists upon an answer to this question, "Don't you see what I am

> showing you?" Naturally, the boy says "yes", and equally naturally his answer is false. The average child does not see what the master is showing him. Froebelianism drives the teacher from words to pictures, from pictures to models, from models to actual objects, and, after all, Herbartianism comes along, and points out that the living sheep that an enterprising schoolmistress has set scampering about the floor of her infant room, does not ensure that teacher and pupil shall speak of the same idea when they talk of a sheep.
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> The popular notion is that knowledge has to be carefully prepared beforehand by the teacher and judiciously stuffed into a suitable place in the pupil's mind, a sort of mental left-luggage office, there to be left till called for. If a mind is not regarded as entirely passive in the process of acquiring knowledge, it is supposed to be active in nothing beyond the stevedore work of lumping the cargo aboard. The mind is assumed to have little power to change a fact that it is acquiring, as a quay labourer to change a granite block he is manipulating. The Herbartian ... has none of that reverence for hard facts, so characteristic of the "plain man". Each soul moulds its own facts, ... every man is his own fact-maker, whether he will or no. (pp. 66–67).

The reader is urged to look beyond the sexist pronouns and illustrations in the above passages – for they were characteristic of 19th-century writing about education – and to reflect on the extent to which some of the Herbartian ideas are similar to those espoused by leading psychologists and educators in the 21st century. The doctrine of apperception, which, it has been claimed (see Henderson, 1911), originated with a mathematician of no less calibre than Leibniz, and was then brought into prominence in education circles by Kant and Herbart, would appear to be especially worthy of closer investigation by mathematics educators of the 21st century. However, as De Vault and Weaver (1970) recognised, latent within the central tenets of the Herbartian movement – based as it was on "the doctrine of interest, the organisation of subject matter around fundamental meanings, and the inclusion of vital and interesting materials in the curriculum" (p. 306) – was a challenge for educators at the beginning of the 20th century. That challenge, which was to construct school mathematics so that it would become more in line with mass elementary and secondary education, has, it could be argued, never been fully grasped.

Ahead of Their Time: What the Herbartians Said About School Mathematics, and 20th Century Mathematics Education Research

We contend that although few 20th century mathematics education researchers became interested in the educational theory of Herbart, or in the modifications of that theory put forward by the Herbartians, in fact elements of the theory slowly moved to centre-stage of the international mathematics education research agenda as the century progressed. We conclude this paper by making eight points intended to illuminate that theme:

1. Like modern-day constructivists, the Herbartians argued that learners constructed their own knowledge. For Herbart (1904a), although the teacher had a vital role to play in facilitating learning, he or she could never be completely sure of what his or her students were learning, or would learn. In the late 20th century, Sinclair (1990) would, likewise, argue that "from a constructivist's point of view, the essential way of knowing the real world is not directly through our world, but first and foremost through our material and/or mental actions" (p. 20).
2. The Herbartians stressed the need to assist students to "chunk" new knowledge so that via the process of apperception new knowledge entities were welcomed into, and integrated with, the learner's "soul". Thus, for example, McMurry and McMurry (1897) provided illustrated lessons showing how students could be assisted to construct general truths through individual notions. Over 100 years later,

Ernst von Glasersfeld (1990) would write: "Good teachers and perceptive cognitive psychologists have always been aware of the fact that what we call knowledge does not enter the uninitiated head in large, complex wholes, but must be built up from components that, all too often, have to be very small elementary pieces" (p. 30).

3. The equilibration aspect of Piagetian theory, by which humans learn through the twin processes of assimilation and accommodation stimulated by cognitive dissonance (Piaget, 1970), was totally anticipated by Herbart's theory of apperception. Nevertheless, it was also the case that the Herbartians' emphasis on the need for the teacher to engineer maximally rich learning environments for students anticipated much of the more social constructivist and interactionist positions of Vygotsky (1978), Lave and Wenger (2001) and Bauersfeld (1995).
4. Steffe (1990) asserted that "in any communication between two human beings, signals can be transmitted between the communicators, but not the intended or received meanings" (p. 7). That was precisely Herbart's (1904b) position. As Adams (1897) argued, the Herbartians had no reverence for hard facts but recognised that "every man is his own fact-maker, whether he will or no" (p. 67).
5. The Herbartians questioned the effectiveness of so-called concrete aids to learning, and emphasised that materials, even live materials, did not assist learning if the materials did not fit the apperceptive needs of the learner (Adams, 1897). During the 20th century, mathematics education researchers (see, e.g., Hiebert & Carpenter, 1992) would, on that issue, reach the same conclusions as had the Herbartians in the 1890s.
6. During the 20th century many cognitive psychologists stressed the importance of prior knowledge so far as the quality and quantity of learning is concerned. For example, Ausubel (1968) asserted that if he had to reduce all educational psychology to just one principle, he would say this: "the most important single factor influencing learning is what the learner already knows (p. vi). In the 19th century Herbart and the Herbartians emphasised exactly the same thing (Adams, 1897). They also emphasised, in their five steps, the need to prepare students mentally for the main idea that a lesson was to be concerned with. In the 1960s Ausubel (1968) would refer to a similar process as "providing an advanced organiser".
7. For Herbart and the Herbartians, teaching had an important role to play in the process of educating moral individuals. The curriculum therefore needed to be carefully planned, and lessons carefully structured and taught so that appropriate learning would occur. To achieve that, the curriculum and the lessons needed to take into account the present social, moral and cognitive states of the learners. That was a hard message for 19th century teachers of mathematics to grasp, because school arithmetic, algebra, and geometry had, for so long, been designed for very capable students who were mainly from well-to-do families. It was well into the 20th century before highly regarded mathematics educators would be prepared to argue that the core part of school mathematics enculturation comes into effect on the meta-level and is learned indirectly (Bauersfeld, 1995), and that children's mathematical constructions are profoundly influenced by social and cultural conditions (Bishop, 1988; Cobb, 1989). Around 1820 Herbart regarded that aspect of his theory as being axiomatic, as did the Herbartians 70 to 80 years later.

8. Wittmann (1998) argued that "mathematics education requires the crossing of boundaries and depends on results and methods of considerably diverse fields, including mathematics, general didactics, pedagogy, sociology, psychology, history of science, and others" (pp. 87–88). According to Wittmann, scientific knowledge about the teaching of mathematics is not gained simply by combining results from these fields, but "presupposes a specific didactic approach that integrates different aspects into a coherent and comprehensive picture of mathematics teaching and learning" (p. 88). Thus, "theoretical studies in the related areas become significant only insofar as they are linked to the core and thus receive a specific meaning" (p. 90). Emphasising the core does not diminish the importance of the related areas.

This emphasis on the need for multiple-perspective studies that combine multiple research methodologies is in line with the Herbartians' view that school curricula should be more integrated or, as De Garmo (1895) termed it, featuring "a close correlation of studies" (p. 217). De Garmo urged a thorough reconceptualisation of the school curriculum, in order "to develop the apperceiving power of the mind" (p. 217). He asked for serious consideration to be given to unifying "all the studies of the elementary school" including arithmetic and any other branches of mathematics. This, De Garmo maintained could "prevent duplication, eliminate non-essentials, and save time and effort" (p. 217). This kind of thinking led Charles McMurry (1906) to expend enormous effort in developing a new "course of study in the eight grades" with the primary aim of facilitating the process by which the child would "develop so that he will respond efficiently to the essential demands of his social environment and of his own individuality" (p, 19).

Towards this end, between 1895 and 1906 McMurry prepared, and had published, separate volumes on special methods in the Reading of English Classics, Primary Reading and Oral Work with Stories, Geography, History, Elementary Science, Arithmetic, Language, and Manual Arts, as well as two volumes that provided an overview of how the curriculum might be correlated. Frank McMurry, Charles' brother, likewise prepared a correlated course of study. For the sixth grade, Frank McMurry's course of study included the following:

> *History*: Causes of the French and Indian War. Desire of France and England to secure the fur trade; differences in religion, etc.
> *Geography*: Valley of St Lawrence, the Great Lakes, Ohio River, Nova Scotia and New Brunswick, Lake Champlain and Lake George, pineries of West and North, fisheries on coast.
> *Science*: Fur-bearing animals – beaver, otter, mink, bear, buffalo, raccoon. Also deer and moose.
> *Arithmetic*: Relative size of the lakes, expressed decimally; of the states in the once disputed territory; relative worth of various kinds of furs, and so forth. (Quoted in De Garmo, 1895, pp. 128 – 129).

Concluding Comments

This "integrated" curriculum position was regarded as extreme by many teachers and teacher educators, and that is one reason why Herbartianism lost ground so rapidly in the 20th century. Another contributing factor was the importance the Herbartians placed on formal teaching, at a time when many wanted school classrooms to be places where children began to assume responsibility for their own learning. A third factor was that Herbartianism came to be associated with normal schools and teachers colleges at a time when recently established faculties of education believed that more "scientific" research, such as that being carried out by E. L. Thorndike, was what was needed in order to place education on a scientific footing (De Vault & Weaver, 1970). Almost certainly, the growth

of anti-German sentiment in the United Kingdom, the United States, and Australia, in the years immediately preceding the first Word War, was another contributing factor.

Be that as it may, throughout the 20th century the ghosts of Herbartianism would have been pleased to observe the master's agenda being increasingly accepted, even if the key players were neither consciously aware of that agenda nor of the history of its elements.

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