LEARNING ABOUT QUADRATICS IN CONTEXT

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This paper discusses one of three case studies to investigate students' conceptualizations of quadratics by solving quadratic contextual problems. The theoretical framework which guided the study was constructivism. The methodology was teaching interviews between a researcher and student solving problems. A multi-representational software, FUNCTION PROBE, was available as a tool to aid students in their problem solving. The results showed that using realistic situations as problem contexts invited multiplicity of interpretations and methods for defining quadratic functional relationships. By reasoning empirically from the problem context, students conceptualized quadratics relationships iteratively and in terms of summation in contrast to the most common view of quadratics as a product of two linear variations. Further, requiring students to verify and justify their strategies by cross-referencing between multiple representations of functional relationships and problem context led them to construct viable schemes to characterize quadratics in terms of rate of change, dimensionality, and symmetry.

Current trends in mathematics education today (NCTM, 1989) recommend that students explore patterns and functional relationships using multiple representational systems (tables, graphs, equations, and diagrams), and examine interconnections between representational systems. The constructivist perspective encourages mathematics educators to focus more on what students say and actively do when solving mathematics problems. (Confrey, (1990a). Researchers have found that a number of students experience great difficulties in understanding the essentials of the function concept (Dreyfus & Eisenburg, 1984; Malik, 1980) and in applying the formal set-theoretic definition to real life examples. (Monk, 1989; Lave, Smith & Butler, 1988). Because of the current practice of introducing the formal definition first before doing real life applications, many students only view functions as a rule of correspondence between domain and range. In contrast, functions as covariance between varying quantities which is more reflective of its historical development and more applicable in realistic situations is often neglected and de-emphasized except in the applied sciences.

THE STUDY

This study was conducted to investigate the use of contextual problems to encourage a developmental understanding of functional relationships in general and quadratics in particular. (Contextual problems refer to problems with context selected from realistic situations that students would encounter in their everyday experiences.) Three high school students and one university student were selected for the study using a pre-test which tested their traditional knowledge of linear and quadratic functions.

THEORETICAL FRAMEWORK

The constructivist perspective based on Piaget's theory of cognitive development was the guiding theoretical framework for this study. In particular, its use of the assimilation and accommodation processes to explain how an individual acquires knowledge. For example, an individual interacts dialectically with the situation or object through reflective abstraction until the individual has come to know the situation or object well and is able to internalize these actions as mental operations. (Kaufman, 1978; Steffe & Cobb, **).

The epistemology of constructivism is based on the philosophy that knowledge is NOT a mirror image of an ontological reality. Instead, knowledge is rooted in human experiences, and is constructed to make sense of human experiences. The starting point is the knower who builds his or her conceptual knowledge solely on the basis of his or her experiences and actions. Human constructions are in terms of schemes or systems of schemes which may be continually re-organized and modified or transformed through social interactions, negotiations of meanings, and communications with others. Hence, within this perspective, mathematics could be viewed as having more than its definitions, theorems, proofs, and its logical relationship, it would also include its form of representations, and its evolution of problems, methods of proof and standards of evidence. (Confrey, 1989).

METHODOLOGY

The methodology was teaching interviews between a researcher and a student solving a contextual problem. Students solved four contextual problems before doing a post-test and a final interview. The type and nature of questions during the teaching interview were guided by students' responses. Each student was given ample time to solve each problem. Students had available to them a multi-representational software: Function Probe (with a table, calculator and graph multi-linked windows) (Confrey et. al, 1989) on the Macintosh for their use a tool in problem solving.

Teaching interviews involved three two-hours sessions per week for six weeks. Each session was audio- and video-taped. The interviewer consistently noted down: (i) students' significant moments particularly when students formulated viable schemes; (ii) cycles of problematics and subsequent (tentative or otherwise) resolutions; (iii) students' rationales for their answers; and (iv) students' verbal and written representations of their ideas.

RESULTS

This paper discusses one of three case studies conducted to investigate students' conceptualizations of quadratic functional relationship by solving contextual problems. The first paper, Afamasaga-Fuatai (1992a), discusses Mary's iterative conceptualizations of quadratics and how she used rates of change (Δy and $\Delta(\Delta y)$) to highlight inadequacies of one algebraic representation to represent her preferred interpretation. In pursuing her preferred interpretation, she examined linear Δy and constant $\Delta(\Delta y)$ values which she later affirmed to be characteristic of quadratic functional relationships. Mary also constructed and further developed a scheme, by reasoning emprically from the context and subsequent numerical patterns, to predict coefficients A and B of a quadratic function: $f(x) = Ax^2 + Bx + C$ in terms of maximum point. (This is similar to the formal formula for calculating the axis of symmetry in terms of A and B.) The third paper (Afamasaga-Fuata'i, 1993) will discuss the case of Bob a senior high school student who conceptualized quadratics as covariations that maximize (or minimize) when the two sets of values: x and y are in a ratio equal to the ratio of their rates of change: Δx and Δy.

In contrast, this second paper will discuss Nan's strategies particularly in how she developed her iterative conceptualization of quadratic relationships to a more elegant and efficient representation as a product of two linear variations. While doing this, she developed a viable scheme for predicting the original quadratic function in terms of rates of change (Δy and $\Delta(\Delta y)$), and like Mary, also characterized quadratics as variations with linear Δy and constant $\Delta(\Delta y)$ ($\neq 0$).

Given the avocado problem as shown below:

> AVOCADO PROBLEM
>
> Farmer Joe has records showing that if 25 avocado trees are planted, then each tree yields 500 avocadoes (on the average). For each additional tree planted, the yield decreases by 10 avocadoes per tree. Determine the number of trees that would maximize total yield.

Nan initially interpreted the loss of fruits conceptually to be the same for all trees. However, in attempting to represent this algebraically, she used an iterative conceptualization that closely reflects her procedural actions of taking off 10 fruits from the newest tree and then 10 more off old trees as generalized by the following:

$s_n = s_{n-1} + (500-10) - 10(n-1)$; for n total trees...(1)

where s_n and s_{n-1} represented current and previous total yields. She explained that this is equivalent to taking 10 off every tree each time a new tree was added. For example:

> "it doesn't really matter if you take ... 30 off 1 tree ... and 20 of another or something ... on average it's all the same once you take care of that ... so that equation still works."

However, after a few more calculations, Nan realized that her iterative strategy was a long and tedious process. Thus, after expressing a preference for a more sophisticated method for predicting maximum total yield, Nan conjectured that perhaps she could utilize the numerical pattern of the differences (Δy) between her iteratively calculated s total yields to predict the original total yields. Nan, however, found that her difference equations were not much help either at this point. (It was not until much later on that these difference equations became significant in predicting the original quadratic function.)

By pursuing a different approach, Nan represented her second conceptualization of total yields as:

$$TY = 12500 + 500(n-25) - 10n(n-25) \qquad (2)$$

in which loss is the same for all trees and is subtracted all at once instead of doing it differentially and iteratively as in her first interpretation represented by s_n. Nan appeared happier with this representation as it allowed her to predict yields for any value of n. Nonetheless, instead of using it to determine maximum yield immediately, Nan chose to reconcile her two interpretations: s_n and TY algebraically as she firmly believed that iteratively subtracting loss of fruits per additional tree had to be the same as taking them all off at the same time. Nan then spent some time manipulating numerical values and equation s_n to effect a reconciliation of the two sets of total yields until she eventually modified s_n to:

$$s_n = 12500 + 500(n-25) - 10(n + (n-1) + (n-2) + \ldots + 26) \qquad (3)$$

Nan also pointed out that the discrepancy between this new form of s_n and the formula TY is with the last term $10n(n-25)$ and $10(n+(n+(n-1)+\ldots+26)$. Nan's problematic then was to represent the summation: $10(n+(n+(n-1)+\ldots+26)$ as a formula. Nan tried but was unable to develop a summation formula. In addition, she found that total yields generated by the modified s_n and formula TY were not the same, instead, the discrepancies increased with increase in trees.

When asked how she would decide the better representation given the problem context, Nan generated average yield values by dividing total yields by trees. Subsequent values showed that average yields from the TY formula (b in Table 1) were more consistent with the loss of 10 in the context than those generated by the iterative s_n. (a in Table 1). Finally, she chose formula TY to be the better representation of the context as shown:

$$s = 12500+500(n-25)-10n(n-25) \qquad (4)$$

Average Yields

When asked for an alternative method of generating average yields, Nan, reasoning empirically from initial conditions, predicted that: average yield = (500-10(n-25)). This new average yield formula was then extended to give an alternative formula for calculating total yields as:

$$\text{Total Yield: } s = n(500-10(n-25)) \qquad (5)$$

Nan immediately commented that this total yield equation was *"more elegant and efficient, and much easier to work with and involved less number of operations"* than her earlier formula TY in (2).

Table 1 - Comparison of Average Yields Using s &TY Strategies.

n	s	a=s/n	ty*	b=ty/n
tree number	total yield	average yield per	adjusted values for	average yield per
25.00	12500.00	500.00	12500.00	500.00
26.00	12740.00	490.00	12740.00	490.00
27.00	12970.00	480.37	12960.00	480.00
28.00	13190.00	471.07	13160.00	470.00
29.00	13400.00	462.07	13340.00	460.00
30.00	13600.00	453.33	13500.00	450.00
31.00	13790.00	444.84	13640.00	440.00
32.00	13970.00	436.56	13760.00	430.00

*ty had formula ty = 12500+500(n-25)-10n(n-25)

First Differences (Δy)
With her more elegant forms of total yield formulas and difference equations for various context variations, Nan was asked to revisit an earlier problematic of developing a scheme to predict total yields using differences. By comparing the new simplified total yield equations in Table 2 below, Nan found that the coefficient of (n-25) term in equation D was consistently twice the coefficient of the n^2 term in the total yield equation as shown:

$\Delta(\Delta y)$ = 2(coefficient of n^2)..............................(6)

Table 2 - Nan's Table With Total Yields in Short Form

Function	Total Yield	Difference Equations	$\Delta(\Delta y)$
1	$TY = -10n^2 + 750n$	D = 240-20(n-25).	-20
2	$TY = -6n^2 + 650n$	D = 344-12(n-25)	-12
3	$TY = -20n^2 + 1000n$	D = -20-40(n-25)	-40

However, while struggling to identify a similar relationship between the constant term in D and n term of TY, Nan noticed that Δy (or D differences) could be generated by using a formula such as:

Difference: $D = \Delta y_1 + \Delta(\Delta y)(n-25)$..............................(7)

where Δy_1 is the first Δy value in the sequence corresponding to the y values of x=0 and x=1, $\Delta(\Delta y)$ is the second difference (difference of Δy), and n represented total trees.

Simplifying her difference equations to: $D_{10}{}^{V}$ = 740-20n compared to $TY_{10} = -10n^2 + 750$ and D_{20} = 980-40n compared to $TY_{20} = -20n^2+1000n$, Nan re-affirmed her relationship in (6), and further pointed out that the

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V The subscript for D and TY, at this point, is to distinguish between the different loss rates. For example, D_{10} and TY_{10} refer to equations for loss of 10. This notation will be used for the loss of 20 and 6 also.

coefficient of the n^2 term would be half of $\Delta(\Delta y)$. After further reflections, Nan noticed that the sum of coefficients of the n^2 and n terms in TY equalled the constant term in equation D. This relationship was further verified with function 2 and its corresponding equations of: $TY_6 = -6n^2+650n$ and $D_6 = 644-12n$.

For discussion purposes, coefficients of n^2 and n terms in total yield equations were labelled A and B respectively to give the general form: $TY = An^2 + B$; and constant term in equation D was E. For example:

$$D = \Delta y_1 + \Delta(\Delta y)(n-25) = [\Delta y_1 - 25\Delta(\Delta y)] + \Delta(\Delta y)n = E + \Delta(\Delta y)n \quad \text{.....(8)}$$

Comparing the two general forms: $TY = An^2 + Bn$ and $D = E + \Delta(\Delta y)n$, and numerical patterns, Nan predicted that: $E = A + B$. where coefficient B would be:

$$B = E - A = E - \Delta(\Delta y)/2 \quad \text{.....(9)}$$

With persistent probing, Nan extended her pattern recognition strategies until she eventually generated <u>two</u> powerful ways of predicting the original total yield equation using difference equations as shown by her schemes in (6) to (9).

DISCUSSION OF RESULTS

Evidently, Nan's initial conceptualizations, developed empirically from the context, had evolved from an iterative sum into a more concise and elegant form as represented by the product of average yields and trees primarily as a result of her own evolving problem solving strategies. Her subsequent interpretations of the problem context and her subsequent representations of her procedural actions led to alternative but viable re-conceptualizations of quadratics as: (i) iterative sums, and (ii) sum of functions in contrast to the common view of quadratics as a product of two linear variations.

1. <u>Quadratics As Iterative Sums</u>. Nan 's iterative conceptualization of total yield was as:

Total Yield	=	Previous Yield	+	Yield from newest tree	-	Loss from old trees	
s_n	=	s_{n-1}	+	(500-10)	-	10(n-1)	.....(10)
or s_n	=	s_{n-1}	+	500	-	10n	.....(11)

In contrast, given the context and intended interpretation of losses from <u>all</u> trees at the <u>same</u> rate of 10 per additional tree per tree and her more elegant total yield formula, (denoted by c for discussion), total yield expressed as a product is: total yield: $c = n * (500-10(n-25))$.

Nan: $s_n = s_{n-1} + 500 - 10n = s_{n-1} + (500 - 10) - 10(n-1)$

26	12500 + 500 - 26(10) =	(500 -10) - 25(10)	12740
27	12740 + 500 - 27(10) =	(500 -10) - 26(10)	12970
28	12970 + 500 - 28(10) =	(500 -10) - 27(10)	13190
29	13190 + 500 - 29(10) =	(500 -10) - 28(10)	13400

Product: $c_n = c_{n-1} + [500 - 10(n-25)] - 10(n-1)$ $\quad c = n(500-10(n-25)$

26	12500 + {500 - 10(1)} - 25(10)	= 26(490)	12740
27	12740 + {500 - 10(2)} - 26(10)	= 27(480)	12960
28	12960 + {500 - 10(3)} - 27(10)	= 28(470)	13160
29	13160 + {500 - 10(4)} - 28(10)	= 29(460)	13340

Figure 3 - A Comparison of Iterative Methods s and c.

To illustrate the legitimacy and viability of the iterative conceptualization of quadratics, Nan's strategy in (10) for the first 4 additional trees is compared to values generated by the re-conceptualized iterative form of equation c in Figure 3 above. Clearly, Nan's iterative s strategy represented a potentially viable alternative conceptualization. With a slight modification however in the interpretation of loss from newest tree, Nan's strategy could parallel that of the iterative c. For example, Nan needs to reflect the variable loss due to the addition of the new tree somewhere in her representation of average yield from newest tree. This could be done either in the loss as in (500-10(n-25)) of the iterative c version, or with the "expected" average yield from tree to tree as in (500 - 10((n-25) - 1) - 10). Since Nan insists on taking "10 off from the newest tree each time a tree was added," she would need to incorporate the variable loss effect on the "expected" average yield while keeping her loss for newest tree at a constant 10 as she preferred.

2. Quadratics As a Sum of Functions. Nan's TY formula suggested a second viable, alternative view of quadratics as a sum of a constant, linear function and a simple quadratic function[1] in contrast to the common product view. For example,

Total Yield = CONSTANT + LINEAR FUNCTION - QUADRATIC FUNCTION ....(13)

Nan's lengthy struggle to reconcile her two strategies (s and TY) was partly because of the arithmetic errors she was carrying throughout her subsequent calculations, and problematic of finding a summation formula.

3. Difference Equations. Nan's most significant construction was her difference-equation scheme which predicted the original quadratic function using rates of change [Δy and $\Delta(\Delta y)$] values. Nan was the only student of the four who represented differences (Δy) algebraically. For example, by expressing the linear Δy values of quadratic variations as a linear function, $D = Mx + E$, she could predict coefficients A and B of the original quadratic function: $y = Ax^2 + Bx + C$. Specifically, $A = M/2$ where $M = \Delta(\Delta y)$ and $B = E - A$. Coefficient C could be determined by finding the difference between predicted values (using only A and B of quadratic function: $y = Ax^2 + Bx$) and actual values as predicted from the context; or, by finding the y-intercept.

USE OF CONTEXT

Data from Nan's case analysis support the use of realistic situations as critical sites for students' mathematizing activities. Nan, given the realistic context and not being aware that it could be modelled by a quadratic function, did not immediately conceive of total yields as the product of average yields and trees. Instead she adopted an iterative strategy of generating total yields. In the course of her struggles with her various problematics and arithmetic errors, she investigated the concepts of rates of change, summations, and multiple representations to a much greater extent than the case would have been in a traditional presentation of quadratic functions. Evidently, solving problems with realistic contexts within a constructivistic perspective can provide students with the opportunity to conjecture and develop their own intuitive conceptualizations into more formalized schemes that are equally viable and even parallel those of formal mathematics. Unquestionably, well chosen realistic contexts is a powerful site for illustrating the inter-relationships of mathematics concepts that is missing from traditional, textbook context-free problems.

THEMES OF QUADRATIC FUNCTIONS

The interviewer initially organized her own conceptualizations of quadratics by using three themes as suggested by Confrey & Smith (1991): (1) rates of change, (2) symmetry, and (3) dimensionality. Briefly, (see Afamasaga-Fuata'i (1992) for a full discussion of this), quadratics functional relationships have linear Δy and constant ($\neq 0$) $\Delta(\Delta y)$ values, line symmetry, and dimensionality factor 2. Nan used all three themes in her characterizations of quadratic functions in contrast to linear ones, and in her various schemes for predicting maximum points, intercepts and original quadratic functions.

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[1]The simple quadratic function is in the form of a product of two linear functions: $y = 10m(m-25)$.

CONCLUSIONS

Nan's iterative conceptualization of quadratics indicated that students' so called errors in problem solving could easily be alternative, rather than erroneous ways of viewing a problem. Knowing and understanding the genesis of students' interpretations, and strategies at a deeper level would provide useful guidelines for the design and development of better contextual problems that: (1) facilitate diversity of interpretations, (2) are challenging enough to motivate students to want to solve them, (3) invite multiple solutions, and (4) embed mathematical concepts that are often difficult to understand by students. The formalization of mathematics concepts, definitions and formulas could develop after students have had a chance to construct their own schemes and conceptualizations of the mathematics embedded in a context.

The interviewer-student interaction eventually led to students' construction of more consistent resolutions and viable schemes. Although this kind of interaction is not viable in a normal class setting with a 1:30 teacher-student ratio, it is, nonetheless, important for identifying schemes students find useful in understanding mathematics concepts; and for considering how to make this kind of interaction occur more in educational settings.

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