KNOWING AND MEANING IN MATHEMATICS CLASSROOMS: PERSPECTIVES DRAWN FROM WITTGENSTEIN'S PHILOSOPHY OF LANGUAGE

CLIVE KANES
Institute for Learning in Mathematics and Language
Griffith University

In discussing Wittgenstein's philosophy of language it is usual to distinguish between his early and late work. His early work, the *Tractatus Logico-Philosphicus* (1921), is famous for its *picture theory* of language. Propositions, he argued, offer us pictures of states of affairs which maintain in reality. They function ostensively, that is by pointing out the facts to which they refer. Later, however, he repudiated this representational view of language, stating

> In the *Tractatus* I was confused about logical analysis and ostensive definition. I thought at that time that there is a 'connection between language and reality'. (Wittgenstein, 1967b)

Wittgenstein's later work, in contrast, focussed on the way language functions as an activity or *language-game*, in which meaning, intention, and purpose cannot be separated out "by a simple formula" from the question of meaning.

This shift in Wittgenstein's view of language from representation to activity, from "purest crystal"(1953/1991, §97) to "blurred edges"(§71) has important lessons, I think, for research in mathematics education

> We have got on to slippery ice where there is no friction and so in a certain sense the conditions are ideal, but also, just because of that, we are unable to walk. We want to walk; so we need *friction*. Back to the rough ground! (§107)

If the objects of theory become too idealised, then ultimately the theory will cease to facilitate practice. Thus, in later Wittgenstein writings great stress is placed on laying bare how language functions, in its "roughness" and diversity.

This paper attempts to apply ideas such as these to mathematics education (*cf* Bloor, 1976, 1983; Confrey, 1981; Hamlyn, 1989; Kanes, 1991a, 1991b; Watson, 1988, 1989). In doing so, the discussion introduces a model (set out immediately below) which attempts to trace out the key relationships between critical elements in any theory of mathematics education: in the first instance these are taken to be variables in the domains of practice (mathematics, learning and teaching) and theory (epistemology, learning theory, pedagogy). Following an analysis of these elements and their interrelationships a third domain of *interior practices*, based on concepts drawn from Wittgenstein's philosophy of language, will be identified and discussed. Note that availability of space has lead to a substantial condensation of the exposition and analysis of the ideas presented here.

SCHOOL MATHEMATICS AND EPISTEMOLOGY

One source of interest in epistemological questions can be related to the collapse of the *process-product* model for conceptualising student/teacher interactions (Putnam, Lampert & Peterson, 1990). In this now discredited model, pedagogy was viewed in purely functional terms: process variables such as teacher proximity, classroom climate *etc* were thought to directly control product variables as measured by student performance in prescribed tasks, affective orientation and so on. In practice, this approach has not proved effective in leading to high quality teacher/learner interactions. Moreover, in criticising this model authors (Romberg and Carpenter, 1986) have questioned whether mathematical knowledge can be validly fragmented in the way commonly practised in research design, instructional design, assessment of the learner's conceptual development and other analytic procedures consistent with this approach. The suggestion is that such epistemological practices are inappropriate for the needs of education in mathematics.

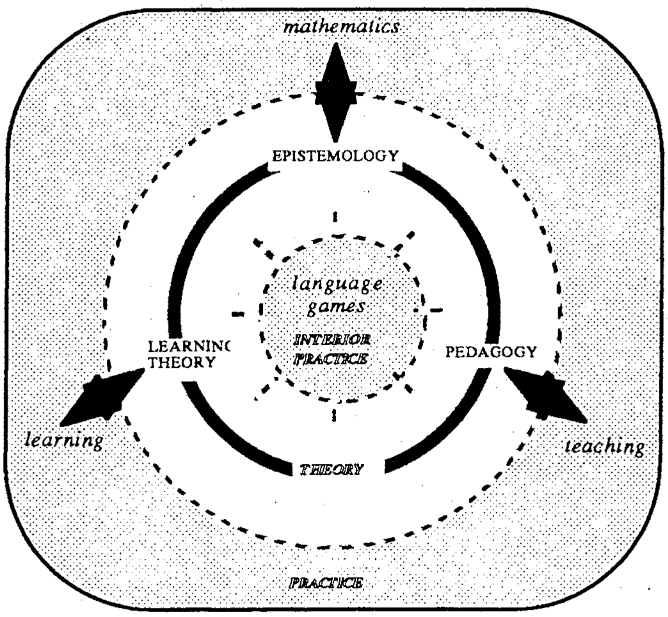

Figure 1: Wittgensteinian model for learning, teaching and epistemology in mathematics education practices

Another source of interest in epistemological questions has been a growing scepticism towards essentialist notions of mathematical knowledge. Relativist views, in which mathematical knowledge is held to be thoroughly social in character (Bloor, 1976, 1983) and dependent on social (Walkerdine, 1988) and cultural (d'Ambrosio, 1985; Bishop, 1988) contexts, are receiving more favoured treatment than in former times. Mathematical knowledge, so it is claimed, is neither culturally neutral nor independent of history. In North America these views have been developed by the so-called radical constructivists (von Glasersfeld, 1984): those who believe the world is a mental construction residing in the knower's mind. Recently, radical constructivist authors such as Paul Cobb (1992) have argued that the neo-pragmatist philosophy of Hilary Putnam and Richard Rorty provides theoretical support for placing mathematics within the sphere of social knowledge.

THEORY AND PRACTICE IN TEACHING AND LEARNING MATHEMATICS

In the literature, there are a number of general approaches to this relationship. An outline of these, including a brief discussion of their relationship to elements of the model, follows.

Reflective practice (Schon, 1983). On this view expert practitioners are skilled in the task of incorporating technical-rationality (theoretical knowledge) within knowing-in-action (expert practice). Schon agrees that the development of the later is enhanced by the process he calls reflection-in-action. This form of knowledge is closely allied to knowledge *of* (see above).

Action research (Carr and Kemmis, 1986). As part of the action research paradigm the teacher is conceptualised as practitioner (both creator of the (lesson)plan and executor of the plan) and researcher (observer and analyser), in a rolling sequence of these roles. Constructs derived from cognitive theory are consistent with this conception. For instance, in Glaser's view (1984) the teacher constructs a pedagogic theory (planning phase) which is similar to, but tellingly different from the content based theory structures held by the students (Carpenter and Peterson, 1988). Action follows (teaching), and this finds students testing, evaluating, and modifying their current theories thereby resolving conflict between these and the teacher's. Observations based on social interactions, including instruments of assessment and evaluation, lead into the fourth phase: teacher analysis and construction of new pedagogic theory depending on teaching goals and the students' conceptual development. The action research approach differs from others in this review in that it is part of the programme to openly treat theoretical constructs explicitly in their propositional form.

Cognitive apprenticeship (Collins, Brown and Newman, 1990). This view is complementary to reflective practice, but contrasts with other approaches in the close proximity established between the teacher and the learner. Emphasis is placed on procedural knowledge. The teacher acts as an exemplary practitioner in the content domain, the learner's knowledge is encoded as site or context specific. Learning is situated in sites of domain practice, thus theory tends to remain implicit, non-propositional in form.

Teacher cognitions and the wisdom of practice. In this research a sharp distinction is made between teacher cognition and teacher actions. Cognitions, including beliefs (Leinhardt, 1990) have been studied using various taxonomies for teacher knowledge (see for example, Shulman, 1987) and include teachers' thoughts during lesson planning and teaching (Peterson, Fennema, Carpenter and Loef, 1989), and beliefs about students, classrooms, and learning. One result to emerge from this research is that, for practitioners, theory derived constructs sit side by side with what Shulman (1986) has called the "wisdom of practice". These distinctions recall the contrasts between Schon's term "technical-rationality" and "knowing-in-action".

Another result is that cognitions and actions are not related in straight forward ways (Clark and Peterson, 1986). This indicates that theoretical constructs are often not, even for the expert practitioner, related "ecologically" (Kagan, 1990) to actual teaching action. Support for Schon's notion of reflective practice in which, for the expert, technical-rationality is related to knowing-in-action, is not therefore confirmed by this body of research.

PARADOXES IN LEARNING THEORY AND PEDAGOGY IN MATHEMATICS

Paradoxes within learning theory and pedagogy and in the relationship between these two, provide a further source for investigating the elements of the model in Figure 1 and their interrelationship. For instance, constructivist learning theory (in which the learner is conceived as being actively engaged in the processes of modifying, transforming, and extending current knowledge structures) although widely accepted (Resnick, 1983; Bereiter, 1985) suffers from what Bereiter has referred to as the learner's paradox. The paradox is that

> if one tries to account for learning by means of mental actions carried out by the learner, then it is necessary to attribute to the learner a prior cognitive structure that is as advanced or complex as the one to be acquired. (p. 202)

Recall that Plato mentions a similar paradox in the Meno (80e): If the student had the knowledge, then there would be no need to seek it, but if the student lacked knowledge, then how would the student even know what to look for? Socrates concludes from this that students can *not* learn what they do not already know. The standard rebuttal of the paradox points to an apparent confusion about the meaning of words, for example, "having knowledge". In Kanes (1991b, 1992) the writer argues that, as with its classical echo, the learning paradox arises from all too narrow views about what constitutes mathematical knowledge. If this is correct, then the significance of the paradox is that it points once again to important weaknesses in the epistemological assumptions upon which cognitive theories of learning and teaching are usually based.

Cobb (1988, 1992) and Kanes (1991b) discuss a related problem, this time concerning the attempt to construct a pedagogy consistent with constructivist learning principles. For instance, talk of "instructional representations" to be used by the teacher in order to prompt the learner's construction of an "internal representation" consistent with the instructional target, is inadequate for constructivist purposes because it leaves unexplained how learners are expected to identify instructional targets within their own internal representations. Once again the underlying issue is epistemological, and the key questions relate to the limits for pedagogical purposes of representational knowledge.

Paradox has also been noted in pedagogy based on more general theoretical foundations. Brousseau (1988) for instance argued that teacher and student enter a *didactic contract* in which the teacher must ensure that the student has an effective means of acquiring knowledge, and in which the student must accept responsibility for learning, even though not being able to see or judge, beforehand, the implication of the choices offered by the teacher. Brousseau argued that the contract is driven into crisis and ultimately fails, for

> all that he [the teacher] undertakes in order to get the pupil to produce the expected patterns of behaviour tends to deprive the pupil of the conditions necessary to comprehend and learn the target notion: if the teacher says what he wants he cannot obtain it. (p.120)

Similarly, Steinbring (1989) observes that the teaching process of making all meanings *explicit*

> leads to the effect that by the total reduction of the new knowledge which is to be learned to knowledge already known, nothing really new can be learned. (p. 25)

Brown, Collins, and Duguid (1989) also emphasise this same point

> Whatever the domain, explication often lifts implicit and possibly even non-conceptual constraints out of the embedding world and tries to make them explicit or conceptual. These now take a place in our ontology and become something more to learn about rather than simply something useful in learning.

Brown *et al* argue that the new construct of situated cognition (sketched below) can afford frameworks for understanding the relationship between explicit knowledge and implicit understanding. Such a theory may provide a better articulation of the epistemological grounding of cognition.

WITTGENSTEIN'S PHILOSOPHY OF LANGUAGE AND PEDAGOGIC INTERACTIONS

For Wittgenstein, human activity is purposeful and thoroughly social in character. Underscoring this is his analysis of language in which the use of words and sentences always form part of an activity. Images of "tools" - language as a tool - and games are therefore recurrent in his later works. Wittgenstein is interested in recovering or demonstrating the multiplicity of meanings carried by ordinary words and sentences: words, on this view, as for poststructuralists, do not have a unitary meaning. Frequently, however, we fall in to the trap of universalising our language-game, failing to notice our grammatical entanglements. According to Wittgenstein,

> We remain unconscious of the prodigious diversity of all the everyday language-games because the clothing of our language makes everything alike. (p.224).

Thus practices of language lead to "conceptual confusion" in ordinary life, in the sciences (*eg* psychology and mathematics, p232), and in philosophy. Conundrums often arise when investigators become overhasty to see the similar in the diverse, and when in an effort to establish this element of similarity they destroy the local character or specificity of particular interactions. As an antidote to this, Wittgenstein suggests it is the task of philosophical analysis simply to make visible the troublesome language-games, those hidden from view by super-familiarity (*cf* §97) and habit. Thus

> Philosophy simply puts everything before us, and neither explains or deduces anything. -Since everything lies open to view there is nothing to explain. For what is hidden, for example, is of no interest to us.(§126)

Consequently, the dictum "Look and see" (§66) applies to Wittgenstein's philosophical method. "One cannot guess how a word functions. One has to *look at* its use." (§340, italics in the original).

Wittgenstein believes it is only in an understanding of the particular purpose in each human action that we are brought closer to understanding language, and through this conceptual knowledge. This contrasts, however, with the view that understanding is generated by working upwards through a hierarchy of ever increasing conceptual generalisations. Wittgenstein assumes we must study activity itself - not propositions or concepts, as in his earlier philosophy - and not activity in general, but the minutiae of specific activities.

Another assumption made is that the study of language and the formulation of language is important because knowledge *eg* mathematical knowledge, is determined by language. This view of mathematics, for instance, implies that it is rooted in social practice. Nevertheless, Wittgenstein maintains a distinction between *facts* which are objects of experience and *concepts* which are social constructs and are therefore the products of language (Wittgenstein, 1967a, p.66e; Wittgenstein, 1953/1991, §364). Metaphysical realism ("Platonism") is denied (as this valorises a notion of 'concept' anterior to activity), as is, for example, Hilbert's formalist program for the foundations of mathematics.

These assumptions combine to explain the significance in his work of the *language-game* construct.

> Here the term "language-game" is meant to bring into prominence the fact that the speaking of a language is part of an activity, or form of life.(§23)

In this same paragraph he provides a list of examples of language-games, or ways in which "symbols", "words", "sentences" are used, these exhibit the scope intended for this concept.

In order to illustrate these ideas, the pedagogic interaction between students and a teacher on the application of 'dummy variables' as indices in expressions involving complex algebraic manipulations will be analysed. The transcript analysed was taken from a video recording of the episode. The normal classroom teacher, Ms X, was not present during the interaction reported. Students are from a Year 11 class in a state run secondary school, located in the southern suburbs of Brisbane.

The transcript set out below is organised against a 'narrative' which, provisionally, divides the interaction into four sub-episodes or 'chapters'. Separate language-games are supposed to correspond with three of these. in the manner shown below. It may be helpful to indicate the purpose of each of these language-games:

- *Language-game of Reference*: Utterances replace (substitute or represent) former symbolic expressions with alternative expressions;

- *Language-game of Action*: Utterances deploy symbols, make them available for action, mathematical procedures;

- *Language-game of Form*: Utterances organise the juxtaposition of symbols and the linkage between terms.

TEACHER/STUDENT(S)	DESCRIPTIVE NARRATIVE
T: Now the very first step here, where you've got $\arg(z_1/z_2)$, Ms X wants you to focus on z_1/z_2. Now the first thing that she did was to write that out in a trigonometric form, or a polar form. And she wrote on the top line, what did she write? Sarah: r_1 ... (inaudible) T: Outside of? Alice: Inside the brackets, I think it's $\cos q_1 + i\sin q_1$ T: Why did she say r_1 and q_1? Ss:(Several students at once) Because that's the modulus and argument for z_1!	<u>Chapter 1: Reference</u> Students and teacher, starting with the language-game of reference, substitute and represent one formulation of symbols with another. This activity is further emphasised by the students' tightly closed representational response to the question "Why did she say r_1 and q_1?"
<u>T</u>: Sorry, just explain? Sorry, who's talking? <u>Alan</u>: Because, well we've got subscript '1', for z_1, we sort of use (*sic*) the same subscript, probably.	<u>Chapter 2: Action</u> The teacher abruptly breaks into a new language-game of action. Actions (explaining, talking) are immediately emphasised. A student responds by spontaneously focussing on how symbols are deployed ("use"), and what facilitates this deployment *ie* the subscript.
<u>T</u>: Would it have mattered what subscript? If (*sic*) she'd written '2', would that have been wrong? (*sic*) If she had written r_2 would that have been wrong?	<u>Chapter 3: Form I</u> The teacher acknowledges student contribution, and henceforth focusses on subscripts. However, emphasis now shifts to an analysis of logical structure.("would that have been wrong?"). Formal structure, linkage between terms (*eg* uses of syllogism) are emphasised. Implications for the overall structure or form of certain symbol deployments is suggested for consideration.
<u>Alice</u>: Only if (*sic*) she had have, it would have been confusing, because you've got z_1 and z_2, and then (*sic*) you've got, it would be easier to have r_1 and q_1 then (*sic*) they've got, it makes a link (*sic*) there, so you have, you say that it's with the same, the same problem (*sic*). <u>Sarah</u>: And also, it linked up (*sic*) to what we did yesterday, because we used z_1 and q_1 and r_1 when we were doing the multiplication as well, so it just tied in (*sic*) with what we did yesterday.	<u>Chapter 4: Form II</u> Students respond by using the language of logical structure (notably syllogism), linkage ("link", "linked"), juxtaposition ("tied in") and identity ("the same, the same problem").

Figure 2: Transcripts

'INTERIOR PRACTICES' WITHIN THE MODEL FOR LEARNING THEORY, PEDAGOGY AND EPISTEMOLOGY IN MATHEMATICS EDUCATION

The writer now turns to consider what implications this analysis has for the model of research and teaching set out earlier in this paper. In this model learning theories, pedagogy, and epistemology are interrelated theoretical domains. Exterior to these, but also in mutual dependence, are the variables of learning/teaching, and school mathematics within the domain of practice. Suggested here is an extension of this model which places language-games as an area of practice embedded within, and therefore interior to theoretical practice. Language-games are therefore thought of as *interior practices*. In summary then, this model consists of three domains distinguished in the following way:

Exterior practices. The domain of practice. What we understand about what we are doing when we teach, learn or perform in a mathematical way.

Theory. In contrast, this domain is about what we understand about our practice *as* teaching, learning or mathematics.

Interior practice. This central domain focusses on what we are *actually* doing when engaged in the activities of exterior practice or theory.

Language games have a generative function. In their activity and practice, larger meanings and purposes, *forms of life*, are generated and these, I think it can be shown, constitute in various ways the theoretical elements of learning, teaching, and knowledge. In what follows two attempts will be made to illustrate the plausibility of these claims.

LANGUAGE-GAMES AND EPISTEMOLOGY

The links between the language-games reference, action, and form (see transcript) and the theoretical domain of epistemology illustrate the point that theories can be built up by the complex interplay of relatively simple language-games. In this case, there is an analogy between Ryle's knowledge *that* (1949) and the language-game of reference signalled here. Propositions match and/or represent actual situations, and thus operate as a representational system. But matching involves referring. We conclude that the linguistic substratum of propositional knowledge belongs to something like the (invented) language-game of reference illustrated in the text above.

Likewise, knowledge *how* is analogous to the language-game of action. Both view words, symbols, gestures, diagrams *etc* as tools (§§23,53) and involve the deployment of these in order to perform activities or functions. (Language-games of reference and form, in contrast, merely displace or juxtapose words and instruments.)

The third category into which the logical objects of knowledge fall is knowledge of persons, places and things. As already indicated, this kind of knowledge, knowledge *of*, is akin to re-cognition (Polyani, 1967; Pring, 1976; Hamlyn, 1970, pp103-111). Since language-games of form involve the articulation of structure and the recognition of juxtaposed elements, there is a strong family resemblance (§§67,77) between the language-games of recognition and those of form.

LANGUAGE-GAMES, LEARNING THEORIES AND PEDAGOGY

A further example in learning theory is situated cognition. Authors (Brown *et al*, 1989; Greeno, 1991; Lave, 1988) propose that the traditional forms of education be shifted from a primary concern for conceptual representation; situated or distributed cognition conceptualises such a shift. According to this theory formal representations of knowledge are indexed by references to context. Thus much of the context is built into or hidden within encoding structures; only the context specific experience of the learner enables knowledge encoded in this way to be utilised. Interior practices provided for here, supply the micro-foundations for these conceptual developments. For example, in the transcript, participating in the language-game of action, Alan constructs a sentence in which he evidently has little faith (hence the terminal "probably") yet nevertheless it is taken as a move in the new language-game of action. How did he construct this utterance? Employment of the word "use" (L16) provides the key. The word is used as an index for the deployment of symbols and words for the purpose of performing a meaningful action. As such, there is a relationship between "use" and similar utterances *eg* "the first thing that she did was to write" (L3), "what did she (ie Ms X) write?" (L6), "explain" (L14), and "talking" (L14) - even where, as in the first two instances, these belonged to different language-games. As part of the original language-game the utterance "the first thing that she did was to write " functioned to highlight the Teacher's reference to the past. This contrasts with the indexical relationship which later becomes established - for here actions are emphasised, and the students deployment of the word "use" repositions the utterance "the first thing that she did was to write" as part of the new language-game of action. In L15-17 Alan has reworded L7-10, however, in doing so he has emphasised the action of setting out the symbols '1', 'z_1', 'r_1', 'q_1' (language-game of action) and down played the activity of (merely) referring to them (language-game of reference). Activity implied by the word "use" is situated in other contexts (another language-game) although it is registered in present actions. This example sketched how language-games become entangled, although remaining self-signifying. The two situations mentioned - situated cognition and the function of indexical representations, and the language-game practices associated with indexical entanglements - are both strikingly similar in correspondence of form and function.

CONCLUSION

The writer has in broad outline shown the connection between interior practices of theory and practice in mathematics education. What implications does this model have for practice? Is the model adequate to provide direction and guidance? The writers answer is, no. Wittgenstein himself did not see philosophical practices as changing the world - on the contrary he claimed that when analysis was correct it was so precisely because it left every thing "just as before". And so it is for the model presented here.

It would be unsatisfactory, however, to leave the matter there. In this it is worth recalling Marx's comment, with his sights set on Hegel's metaphysics, that hitherto philosophy had set out to explain the world, however, the task is to change it. It may be true that in "setting the world before us" Wittgenstein's work threatens to become mere description. If so, then parallels between Wittgenstein and Hegel may be found. In (almost all) other respects, however, these two philosophers take contrary positions for, as Wittgenstein himself pointedly remarks, Hegel works to recover unity in the diversity of phenomena, whereas the theory of language-games pulls the other way - it is interested in showing that often what we are inclined to think of as being one (mathematical knowledge, for instance) is really a multiplicity of self referring practices or activities. Wittgenstein argues that in the function of language, we cannot rely on the crystal clarity of unambiguous distinctions. The edges, he maintains, are blurred and *must* be blurred. Unity gives way to diversity.

Another, but related difficulty with Wittgenstein is that, again like Hegel and unlike Marx, his work proceeds without reference to concepts of power, in which political and economic issues arise. For instance, the relative status of teacher and students, participants in the construction of interior parameters (language-games), and the relationship of these to traditional practices, the material conditions of work, executive authority, policy, and the state, are not considered.

It is clear that in order to join the diversity spoken of in Wittgenstein's analysis, with an analysis of the constraints just referred to, considerable broadening of the conceptual structures available for theory building would need to be facilitated. Foucault's notion of *discourse* as a practice that systematically forms the objects of which it speaks, may be the construct needed here. For Foucault, it is the extra-representational function of a discourse which renders it significant for any theory of knowledge and society.

> discourses are composed of signs; but what they do is more than use these signs to designate things. It is this 'more' that renders them irreducible to language (*langue*) and to speech. It is this 'more' that we must reveal and describe. (Foucault, 1972, p. 49)

The description of this 'more' may be no more urgent in any subject domain, than in school mathematics.

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