Semantic Characteristics That Make Arithmetic Word Problems Difficult

Ban-Har Yeap
Nanyang Technological University, Singapore
<bhyeap@nie.edu.sg>

Berinderjeet Kaur
Nanyang Technological University, Singapore
<bkaur@nie.edu.sg>

This paper describes an investigation on semantic characteristics that make an arithmetic word problem difficult for children. Five semantic characteristics were delineated for investigation. They are (1) type of semantic relations, (2) number of semantic relations, (3) number of types of semantic relations, (4) presence of simultaneous unknowns, and (5) nature of unknowns. 436 year three and 116 year five children were given pairs of word problems that differ in one of the five semantic characteristics. It was found that the presence more types of semantic relations and of simultaneous unknowns significantly increases the difficulty of arithmetic word problems.

Word problems have long constituted a major part of elementary school mathematics. A variety of reasons have been put forth to justify this privileged position. Verschaffel, Greer and De Corte (2000) presented a list of such reasons which includes providing opportunities for students to use mathematical tools, motivating the link between mathematics and real-life context, encouraging thinking and the use of problem solving heuristics, and providing a platform to develop new concepts and skills. Several critics (Lave, 1992; Gerofsky, 1996) have questioned such justifications. Gravemeijer (1997) has even suggested that word problems are often mere disguise for practice in the four basic operations.

Verschaffel, Greer, and De Corte (2000) have called for the re-conceptualisation of the role of word problems in school mathematics. They have suggested the use of word problems to engage students in mathematical modelling. In other words, students solving word problems should engage more in making sense of the semantics of the problem and less in doing tedious computations.

Early research on word problems focused on superficial characteristics such as the length of problems and the placement of questions. In the late 1970s, semantic characteristics of word problem began to receive attention (Riley, Greeno & Heller, 1983). Addition and subtraction word problems are classified into three main categories: Change, Combine, and Compare. Carpenter and Moses (1982), Nesher (1982), Vergnaud (1982), and Fuson (1992) have subsequently proposed and extended similar classification schemes. Later, Greer (1992), Nesher (1988), and Kaput (as cited in Greer, 1992) proposed classification schemes for multiplication and division word problems. One common feature of all these schemes is that the schemes are for one-step word problems and do not provide for comparison between problems with additive structures and those with multiplicative structures.

Marshall (1995) provides an alternative classification scheme for arithmetic word problems that is situation-based and not operation-based. Her categories of Group (the same as Combine in earlier schemes), Change, Restate (the same as Compare in earlier schemes), Vary, and Compare (not found in earlier schemes) are suitable for analysis of multi-step word problems.

Semantic Characteristics

Semantic relations refer to the relationships between sets of physical quantities. Marshall (1995) has previously identified a basis set of semantic relations in arithmetic word problems. Such a basis set comprises five distinct types of semantic relations. The five semantic relations are named *Change, Group, Restate, Vary,* and *Compare*. In the Change relation, there is an initial state, a change and a final state. There are three values in such a relation, the magnitude of a physical quantity at the initial state, the magnitude of the change and the magnitude of the physical quantity at the final state. In the Group relation, there are sub-sets that form a set. There are at least three values in such a relation, the magnitude of the physical quantity in the sub-sets as well as the magnitude of the physical quantity in the set. In the Restate relation, there is a specific relationship between two sets of physical quantities at a given point in time. There is a quantity that is described in both absolute and relative terms. In a Vary relation, there is a specific relationship between two sets of physical quantities. This relationship is maintained when the magnitude of either quantity changes. In a Compare relation, two or more sets of physical quantities are compared to determine which is larger. Table 1 provides examples of word problems that contain the five semantic relations defined.

Table 1
Examples of Word Problems with Each Semantic Relation

Semantic relation	Example
Change	Chris had 623 stamps. He gave his friend 572 stamps. How many stamps does Chris have now?
Group	Ginny baked 315 chocolate cookies. She also baked 59 vanilla cookies. How many cookies did she bake?
Restate	Royston has 316 marbles. Ray has 49 marbles more than Royston. How many marbles does Ray have?
Vary	There are 25 cookies altogether. Valerie puts them into 5 boxes. How many cookies are there in each box?
Compare	Mason has 120 stamps. Molly has 115 stamps. Who has more stamps?

Word problems with the same semantic relations may differ because the *nature of unknowns* is different. In a Group relation, there are at least three values, the magnitude of the physical quantity in the sub-sets as well as the magnitude of the physical quantity in the set. In such word problems, any two may be given and the third is unknown. In a Change relation, the unknown can either be the initial value, the change value or the final value. In a Restate relation, the unknown can either be the absolute value or the relative value. In the example shown on Table 1, the relative value of the number of marbles Ray has ('49 marbles more than Royston') is known but its absolute value is not. In a Vary relation, the unknown can differ such that the problem either involves multiplication, partition division or quotition division. Table 2 provides examples of the three Vary problems that differ in this semantic characteristic.

Table 2
Three Types of Vary Word Problems

Example	Semantic relation	Nature of the unknown
Each box of pencils contains 12 pencils. There are 6 boxes of pencils. How many pencils are there altogether?	Vary	1 part = 12 6 parts = ? Multiplicative
There are 95 cookies altogether. Peter puts them into 5 boxes. How many cookies are there in each box?	Vary	1 part = ? 5 parts = 95 Partition division
Miss Quah groups her pupils in groups of 4. There are 36 pupils. How many groups are there altogether?	Vary	1 part = 4 ? parts = 36 Quotition division

Many word problems comprise more than one semantic relation. Table 3 shows two such examples. In the problem Hall, there are three Group relations (boys and girls, men and women, children and adult). In the problem Toys, there are three relations - one Restate (8 more toy trucks than toy cars), one Group (trucks and cars) and one Vary (toys per box). The number of semantic relations is three in both problems. However the *number of types of semantic relations* is one in the former and three in the latter.

Table 3
Single-Relation and Multi-Relation Word Problems

Semantic relations	Example
Group Group Group	Hall: There are 45 boys and 68 girls in a hall. There are also 10 men and 20 women. How many people are there?
Restate Group Vary	Toys: David has 23 toy cars. He has 8 more toy trucks than toy cars. He puts these toys equally into 9 boxes. How many toys are there in each box?

The number of semantic relations and the number of types of semantic relations in two word problems may be identical but another semantic characteristic may differ. This characteristic is the *presence or absence of simultaneous unknowns.* In the problems Azlan and Betty, shown in Table 4, the number and types of semantic relations are the same. The semantic relations are Restate ('4 marbles more than Azlan' in Azlan and '$20 more than

Bala' in Betty) and Group ('How many marbles do they have?' in Azlan and 'Betty and Bala have $300 altogether' in Betty). They, however, differ in the presence or absence of simultaneous unknown. In Azlan, the number of Azlan's marbles is given and it is known that Amy has 4 marbles more than Azlan. Hence, one of the unknowns - the number of Amy's marble – can be found before the second unknown – the number of Azlan's and Amy's marbles – is found. There is only one unknown at every stage. There is no simultaneous unknown. In Betty, although the amount Betty and Bala have is known, the amount each has is unknown. Although it is given that Betty has $20 more than Bala, neither the amount Betty has nor the amount Bala has is known. There are two simultaneous unknowns.

Table 4
Problems With and Without Simultaneous Unknowns

Semantic relations	Simultaneous unknowns	Example
Restate Group	Absent	Azlan: Azlan has 36 marbles. Amy has 4 marbles more than Azlan. How many marbles do they have?
Restate Group	Present	Betty: Betty and Bala have $300 altogether. Betty has $20 more than Bala. How much does Bala have?

Purpose

The main purpose of the present investigation was to determine what semantic characteristics in word problems make them difficult for children to solve. The specific research questions are:

- Is a word problem with one type of semantic relation more difficult than another with a different semantic relation?
- Does the difficulty of a word problem depend on the nature of the unknowns?
- When word problems have more than one semantic relation, is a word problem with more semantic relations more difficult than another with fewer semantic relations?
- When word problems have more than one semantic relation, is a word problem with more types of semantic relations more difficult than another with fewer types of semantic relations?
- Is a word problem with simultaneous unknowns more difficult than another without simultaneous unknown?

Method

Subjects

The subjects comprised 436 year three and 116 year five children in three schools. The number of boys was approximately equal to the number of girls. Every child in the twelve year three classes and three year five classes who were present on the day of data collection took the word problem test. The heads of mathematics department in the three schools were asked to use their professional judgment to identify classes that were high-achieving, mid-achieving and low-achieving. There were four primary three classes and a primary five class in each achievement category.

Procedure

Pairs of arithmetic word problems were written such that one of the five semantic characteristics differed. Factors such as number of words, vocabulary, number type and magnitude of numbers were controlled. These pairs of word problems were distributed into five forms. Each form typically contained ten word problems, including dummy problems. In each form, the order of the problems was systematically varied to counter the effect of order. Three versions of each form were consequently generated.

Each form was assigned to one high-achieving class, one mid-achieving class and one low-achieving class in either year three or year five. About 110 children attempted each of the five forms. The children's respective mathematics teachers conducted the paper-and-pencil test. The children were allowed one hour to complete the test.

Findings and Conclusions

Each response was coded as successful (1) or unsuccessful (0). A successful response included solution that used a correct method but had some computation mistake. The data were cross-tabulated and a statistical test for categorical data (McNemar's Test) was used to determine if a problem was significantly more difficult to solve than another that differed in one of the five semantic characteristics.

Is a word problem with one type of semantic relation more difficult than another with a different semantic relation? A problem with each semantic relation was compared to problems with every other semantic relation. There was insufficient evidence to reject the null hypothesis at a significance level of 0.05 that word problem with one semantic relation was as difficult as one with another in all except two comparisons. Restate problem and Compare problem were significantly more difficult than Group problem to the children.

Does the difficulty of word problems depend on the nature of the unknowns? For Group problems, Change problems and Restate problems, children did not find those with one type of unknowns more difficult than one with another. For example, a Change problem with the unknown initial amount was not more difficult than one with an unknown final amount. Restate problem with unknown relative quantity was as difficult as one with unknown absolute quantity. Among the Vary problems, quotition division problems were found to be more difficult than the other two types. The difference was, however, significant only between quotition division problems and multiplication problems. There difference was not significant between the two types of division problems.

When word problems have more than one semantic relation, is a word problem with more semantic relations more difficult than another with fewer semantic relations? Word problem with two Group relations was compared to a problem with one Group relation. Also, word problem with two Restate relations was compared to a problem with one Restate relation. In each comparison, there was insufficient evidence at a significance level of 0.05 to reject the null hypothesis that a problem with more than one semantic relation was as difficult as a problem with one semantic relation, given that the type of semantic relation remained the same.

When word problems have more than one semantic relation, is a word problem with more types of semantic relations more difficult than another with fewer types of semantic relations? The relative difficulty of problems with different number of semantic relation types was established by comparing the difficulty of problems with one type of semantic relation (Restate/Restate and Group/Group/Group) with those with two types (Restate/Group and Restate/Restate/Group) and three types (Restate/Group/Vary) of semantic relations respectively. The former is called *single-relation problem* while the latter is called *multi-relation problem.* The relative difficulty of problems with two and three types of semantic relation was also compared. There was sufficient evidence at a significance level of 0.05 to reject the null hypothesis that multi-relation problems were as difficult as single-relation problems. There was, however, insufficient evidence that a problem with three types of semantic relation was more difficult than one with two types of semantic relation. In other words, multi-relation problems were of the same difficulty level.

Is a word problem with simultaneous unknowns more difficult than another without simultaneous unknown? Three pairs of problems were compared. In each pair, the problems have the same number of type of relations but in one there is simultaneous unknowns while in the other there is no simultaneous unknowns. Table 5 shows one of the problem pairs.

Table 5
Problems With and Without Simultaneous Unknowns

Subjects	Semantic relations	Problem without simultaneous unknowns	Problem with simultaneous unknowns
Year 3	Restate Group	Marbles: Mary has 36 marbles. John has 3 marbles more than Mary. How many marbles do they have?	Money: Ahmad and Raju have $300 altogether. Ahmad has $20 more than Raju. How much does Raju have?

It was found that while between at least 70% (and as high as 80%) of the children were successful with problems without simultaneous unknowns, at most 40% (and as low as 24%) of them were successful with those with simultaneous unknown. There was sufficient evidence to conclude that problems with simultaneous unknowns were more difficult than those without even among older children in year five.

The present investigation aimed to identify semantic characteristics that make arithmetic word problems difficult to solve. It was found that word problems with simultaneous unknowns were more difficult than those without. The traditional heuristic of choosing an operation would be less successful in solving problems with simultaneous unknown. Yeap and Kaur (2001) have observed children who were successful with these problems using heuristics more sophisticated than the choosing an operation heuristic. It was also found that multi-relation word problems were more difficult than single-relation problems. A single-relation problem that involves just as many computation steps as a multi-relation problem is easier to solve because any difficulty is not a relational one but an instrumental one (Skemp, 1978).

Among single-relation problems, Restate and Compare problems were more difficult than Group problems. Previous research among young children has established that Restate problems are difficult compared to Group and Change problems (Riley, Greeno, & Heller, 1983; Verschaffel & De Corte, 1999). Compare problems are probably difficult because of unfamiliarity rather than semantic complexity. Few word problems in textbooks that the children use include Compare relation. Among Vary problems, those that involve quotition division were found to be more difficult. This is consistent with the theory that partition model is the primitive intuitive model for division while quotition model is acquired later through instruction (Fischbein, Deri, Nello, & Marino, 1985). English (1996) has also found that year five children tend to pose partition problems rather than quotition problem when given a division sentence.

Applications of the Findings

The present investigation delineated five semantic characteristics of arithmetic word problems and found that certain characteristics make word problems difficult. This section briefly outlines two subsequent investigations that use the findings of the present investigation.

In the first investigation, knowledge of semantic characteristics is used to select and vary word problems in instructional materials. The use of variants in semantic characteristics as well as their possible contribution to problem difficulty allows children to be exposed to a wide variety of word problems appropriately. The opportunity to solve a wide range of problems with different semantic characteristics will require children to make sense of the context and meaning in word problems. This will hopefully able to prevent the phenomena documented by Verschaffel, Greer, and De Corte (2000) where children seemed to suspend their ability to make sense when they solved word problems. In this investigation, the main purpose is to facilitate thinking and sense-making through word problems.

In the second investigation, the findings are used to develop a framework to analyse problem posed by children. In this framework, word problems posed by children are assigned an index. The framework facilitates a two-level analysis of word problems posed by children. At the first level, a problem is coded as either solvable or non-solvable. At the second level, a solvable word problem is assigned a higher index if it is more difficult to the peers of the problem-poser. Thus, a multi-relation problem as well as one with simultaneous unknown are assigned higher index than a single-relation problem or one without simultaneous unknown. This index is then used as a measure of mathematical problem-posing ability amongst children.

References

Carpenter, T. P., & Moser, J. M. (1982). The development of addition and subtraction problem solving skills. In T. P. Carpenter, J. M. Moser, & T. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 9-24). Hillsdale, NJ: Lawrence Erlbaum.

English, L. D. (1996). Children’s problem posing and problem solving preferences. In J. Mulligan & M. Mitchelmore (Eds.), *Children’s number learning* (pp. 227-242). Adelaide: Australian Association of Mathematics Teachers.

Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16,* 3-17.

Fuson, K. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 243-275). New York: Macmillan.

Gerofsky, S. (1996). A linguistic and narrative view of word problems in mathematics education. *For the Learning of Mathematics, 16*(2), 36-45.

Gravemeijer, K. (1997). Solving word problems: A case of modeling? *Learning and Instruction, 7,* 389-397.

Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 276-295). New York: Macmillan.

Lave, J. (1992). Word problems: A micrososm of theories of learning. In P. Light & G. Butterworth (Eds.), *Context and cognition: Ways of learning and knowing* (pp. 74-92). New York: Harvester Wheatsheaf.

Marshall, S. P. (1995). *Schemas in problem solving.* New York: Cambridge University Press.

Nesher, P. (1982). Levels of description in the analysis of addition and subtraction. In T. P. Carpenter, J. M. Moser, & T. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 25-38). Hillsdale, NJ: Lawrence Erlbaum.

Nesher, P. (1988). Multiplicative school word problems. Theoretical approaches and empirical findings. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 19-40). Hillsdale, NJ: Lawrence Erlbaum.

Riley, M. S., Greeno, J. G., & Heller, J. J. (1983). Development of children’s problem solving ability in arithmetic. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 153-196). New York: Academy Press.

Skemp, R. R. (1978). Relational understanding and instrumental understanding. *Arithmetic Teacher, 26*(3), 9-15.

Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction. In T. P. Carpenter, J. M. Moser, & T. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 39-59). Hillsdale, NJ: Lawrence Erlbaum.

Verschaffel, L., & De Corte, E. (1999). Word problems: A vehicle for promoting authentic mathematical understanding and problem solving in the primary school? In T. Nunes, & P. Bryant (Eds.), *Learning and teaching mathematics: An international perspective* (pp. 69-97). United Kingdom: Psychology Press.

Verschaffel, L., Greer, B., & De Corte, E. (2000). *Making sense of word problems.* Lisse, The Netherlands: Swets & Zeitlinger B. V.

Yeap, B. H., & Kaur, B. (2001, March). *Mathematics problems children in Singapore solve*. Paper presented at the First SEAMEO Educational Congress, Bangkok, Thailand.