Teaching for Abstraction: Angle as a Case in Point[1]

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The method of Teaching for Abstraction was applied in Grades 2 and 4 to individual teaching interviews focussing on the angle concept. Responses of 3 students who successfully matched different physical angle situations are compared to responses of 3 that did not. The teaching method was generally supported, but it appears that more emphasis needs to be given to the examination of within-situation and within-context similarities before matching across contexts, and hence abstraction of the angle concept, can proceed.

School students have great difficulty learning the angle concept (Clements & Battista, 1992). Douek (1998), for example, described how difficult it was for students in Grades 3 and 4 to interpret the inclination of the sun in terms of angles. The problem seems to be that angle is such a multifaceted concept. Close (1982) and Krainer (1989) have discussed the variety of angle definitions used in mathematics, and many authors have noted the difference between dynamic (movement) and static (configurational) aspects of the concept (Close, 1982; Kieran, 1986).

Mitchelmore & White (2000a) have recently attempted to explain the formation and integration of the various facets of angle into a single concept by calling on the theory of abstraction and generalisation. According to this theory, an abstract concept is "the end-product of ... an activity by which we become aware of similarities ... among our experiences" (Skemp, 1986, p. 21) and generalisation is the process which extends the meaning of such a concept to include further experiences. Mitchelmore and White propose that children first recognise superficial similarities between angle experiences and form separate angle concepts based on *physical angle situations* such as tiles, hills, and wheels. Then, they recognise deeper similarities between these situations and form angle concepts related to *physical angle contexts* such as corner, slope, and turn. Finally, they recognise even deeper similarities between contexts and form an *abstract angle concept* which gradually generalises to include all angle contexts.

The empirical results reported by Mitchelmore & White (2000a) show that children rarely recognise the dynamic similarity between two physical angle situations. Students more easily recognise the static similarity, the facility depending on the salience of the two lines which form the angle in each context and increasing with age. Over 90% of the Year 2 children in their sample readily recognised the angular similarity between situations such as tile corners, road junctions, and scissors where the two lines are obvious, but it was not until Year 6 that a similar percentage included hand fans and leaning signposts, where the two lines are less obvious, in this category. Students found it even more difficult to recognise the angular similarity to situations where one or both lines has to be imagined (hills, doors, and wheels); even in Year 8, about one-third of students could not identify the two arms of the angle implied in these situations.

Based on these findings, Mitchelmore and White (2000a) suggested the following sequence for teaching angles in primary school:

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1. Develop an agreed language (both graphic and verbal) for describing various corner situations where both lines of the angle are clearly visible
2. Using this language, explore the similarity between such situations and corner situations where both lines of the angle are visible but not clearly defined.
3. Explore the similarity between angle situations where both arms of the angle are visible and angle situations where only one line of the angle is visible.
4. Explore the similarity between angle situations where both arms of the angle are visible and angle situations where no line of the angle is visible.

In Stage 1, the teacher would help children to abstract an angle concept from a similarity they probably already recognise. In subsequent stages, the teacher would help children to recognise the angular similarity between the "old" and the "new" situations and hence to generalise their angle concept. The procedure is an example of Teaching for Abstraction (Mitchelmore & White, 2000b).

Selective attention influences the way children develop their angle concepts (Owens, 1998). Abstraction should therefore be enhanced if the teacher focuses children's attention on the common features of the abstract angle concept (two lines, a vertex, and an opening). If these three features are clearly identified at Stage 1, their identification in the "new" situations in Stages 2-4 should assist children to recognise the angular similarity and hence to generalise their angle concept. Students' attention may be drawn to the critical features of an angle in various ways: (a) by questioning them about the similarities they recognise; (b) by having them indicate the angles with a bent straw or similar abstract angle model; and (c) by asking them to draw one diagram to represent the common features.

Some preliminary findings (largely quantitative) from an exploratory investigation of the proposed teaching model are reported in White and Mitchelmore (2001). This paper provides a more in-depth look at some of the qualitative data.

Method

Participants were 19 Grade 2 and 20 Grade 4 students, balanced for gender, randomly selected from 2 classes in a school in Sydney. A trained research assistant[2] taught students in individual interviews lasting about 40 minutes. The materials used were models of 8 physical angle situations: (a) three walls intersecting at 90°, 67.5°, and 45°; (b) three rhomboidal tiles with corners of 90°, 67.5°, and 45°; (c) a pair of scissors; (d) a hand fan; (e) a hinged door; (f) a tilting window; (g) a wheel; and (h) a regulator knob. For each stage of the teaching sequence outlined above, the interviewer presented students with various pairs of these models: (1) tile-wall; (2) scissors-tile, scissors-fan; (3) scissors-door, door-tile, door-window; (4) wheel-scissors, wheel-knob. The teaching procedure for the first pair, tile-wall, was as follows:

- The interviewer first pointed out how the three tiles were different and how the three walls were different. He then asked students to match each tile to one of the walls. If students had any difficulty in doing this, he asked them to say what made the three tiles different and what made the walls different. If students still could not match the tiles and the walls, he showed one match and asked students to match the others. If this was unsuccessful, he showed all three matches.

[2] We are grateful to Michael Cavanagh for his sensitive interviewing and constructive contribution to the data analysis.

- Next, the interviewer asked or helped the students to identify exactly what features of the tiles and walls matched, and then guided them to identify the three critical features. He used a bent straw when necessary to draw students' attention to the two lines, the vertex, and the opening.
- Students were finally asked or helped to draw one picture to represent each matching wall-tile pair. Again, a bent straw was used when necessary. For each diagram, students were asked (a) to indicate how the lines and the vertex matched parts of the wall and the tile, and (b) to demonstrate how the diagram showed the correct amount of opening.

The teaching procedure for subsequent pairs of situations was basically the same except that the interviewer helped students where necessary to identify the lines, vertex, and opening in each situation. In all cases, the interviewer first asked students to perform each task unaided; if they were unsuccessful, he provided help by focussing their attention on the critical features; and if this was not successful, he demonstrated an appropriate solution. All the teaching interviews were videotaped.

Results

The experimental teaching procedure appeared to help some students recognise angle-related similarities, but for many students it was not successful. In an attempt to identify the crucial aspects of the teaching method which led to success, we compare below the responses of three students who were clearly successful with the responses of three students who were not successful. Since the reponses of Year 2 and Year 4 students were in many ways similar, we do not distinguish between the two age groups.

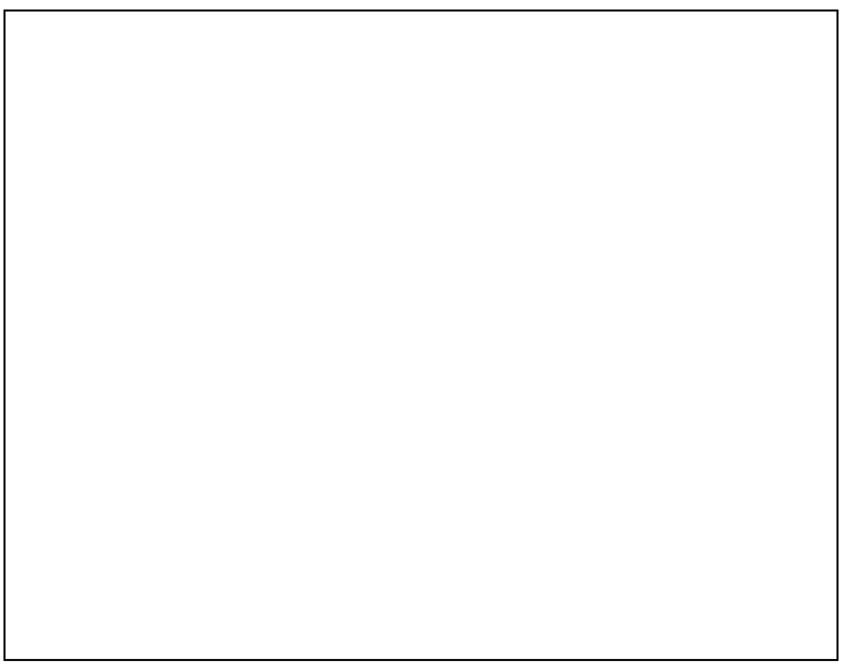

Figure 1. Examples of irrelevant matching between angle situations.

Unsuccessful or Doubtful Matching

When asked to match pairs of situations, students often focussed on irrelevant attributes, as shown in Figure 1. The following excerpts illustrate cases where Teaching for Abstraction appears to have failed to direct students to the relevant attributes.

Student 1. This student had previously matched the angles in the tile-wall, tile-scissors, and scissors-fan pairs in Stages 1 and 2. Then she was asked to match the angle in the door with the angle in the scissors. The interviewer (I) showed how the door opened and closed and then asked the student (S) if she could open the scissors to the same amount.

> S: Which angle are you talking about - there are 4 [points to the four corners of the open door.] I: Well they're points aren't they, you pointed to the corners. Can you see the angle in the door opening [opens and closes the door again.] S: Oh - is it there?[S runs her finger up the edge of the door.] I: Can you use the straw to show me the angle? [S places straw on the corner of the door as shown by the scissors in Figure 1(b).] Is that where the angle in the door is? S: No. not really. I: Because it can go from no angle to a small angle to a bigger angle [I closes door, then opens it a small amount and finally opens it a larger amount.] Where you've placed the straw, it stays the same no matter where the door is. [S now places the straw correctly in the opening of the door.] Good, now can you match the scissors to the door. [S places scissors at the base of the door, but has difficulty manipulating them.] You can match them on top if it's too difficult to move them around down there. [S places scissors where she had previously placed the straw - position shown in Figure 1(b).] Think about the angle you showed me with the straw and try to match the scissors to that angle. [S places the scissors at the base of the doorway again, but matches the end points as shown in Figure 1(c).]

This student was unsuccessful in matching the dynamic angles in the scissors and the door situations because she focussed on corners (probably connecting with the previous examples). Even though she could clearly identify the angle in the scissors, irrelevant matchings with the door indicate that she could not interpret the opening of the door in terms of angles. Furthermore, even after the student had successfully learned to match the bent straw to the relevant door angle, she still could not match the scissors to the door opening.

Student 2. This student had previously matched the angles in the tile-wall and tile-scissors pairs. Then he was asked to match the angle in the scissors with the angle in the fan. The interviewer showed how the fan opened and closed, left it open at about 80°, and then asked the student if he could open the scissors to the same amount. The student opened the scissors and placed them on the fan as shown in Figure 1(a).

> I: That's where the scissors are opening and closing, where's the fan opening and closing from? [S points randomly at the fan.] When I open the fan [I picks it up to demonstrate], where's the fan opening and closing from? [S points to the edge of the fan where I held it when opening it.] Here is the scissors opening and closing and here is the fan opening and closing. You're trying to match the sides on the scissors with the sides of the fan and the point of the scissors with the point of the fan. [I indicates the edge of the fan.] Where do you think that would be on the scissors. S: It's the side. I: Good. Where is the point on the scissors? [S points to the vertex of the scissors.] Where's the same thing on the fan? [S points to the end point of one edge of the fan].

The student continued to have difficulty identifying the vertex in the fan, so the interviewer eventually showed him the lines and point in the scissors and then the lines and points in the fan. The student then placed the scissors correctly on the fan and was able to repeat the match for various other openings.

The student then drew an inverted V-shape to represent the angle in both the scissors and the fan, correctly matching the points and lines in both situations with the corresponding parts in the diagram. However, when asked to repeat the exercise for the scissors and fan opened to bigger angles, the student drew an angle which was the same size as before but with longer arms.

> I: How could you show me if your drawing is the right size? [S superimposes the open scissors on top of the drawing.] Does that look the right size? S: No. I: Is your drawing too big or too small? S: Too small. I: Do you think you could try and draw it so that it is the right size? [S draws another same size angle immediately above the one he just checked.] Do you think you could use the scissors to help? S: Hmm. [S takes scissors over to the page, places the vertex on the vertex in the drawing and traces out the angle.] I: How does that look? S: Good.

The student's inability to identify the vertex in the fan made identifying the angle in the fan impossible. When assisted with this identification, he could match the key components of the angle in the scissors and the fan. However, the student's attempts at drawing the common features indicated that he was not identifying the opening aspect of the angle; his apparent success in superimposing the scissors on the fan was a result of his matching only the vertex and the sides of the angle.

Student 3. Like Student 1, this student has difficulty matching the scissors and the door. She indicated the match in Figure 1(c).

> I: Think about the things you are trying to match. Remember the things you have to match? S: Yes. [S places scissors on the floor in the doorway, but on the opposite side to the door.] I: Are the lines on the scissors matching the lines on the doorway? [S looks confused.] Where do you think the point is on the door? [S points firstly to the outer corner on the base of the door and then correctly to the inner corner.] Can you run your finger along the lines in the door? [S runs his finger along the bottom edge of the door and the imaginary line in the opening.] Can you now match the scissors with the door? [S tries to do so at the bottom of the door and has some trouble.] S: The handles are in the way. I: You could try it on top. [S makes a correct match.]

This student appears to successfully identify the angular attributes in both situations, but we view his success as ambiguous because at no time is the opening in the door matched with the opening in the scissors. In fact, when asked to say why the situations matched, the student (in common with many other students) only commented on the lines and vertex. It may well be that this student was also only matching lines and points and not angles as such

Successful Matching

Student 4. The student was presented with the scissors and the door after successfully completing Stages 1 and 2.

> I: Can you show me how the scissors and the door match? [S places the scissors as shown in Figure 1(c).] Good, yes, that's one way of matching them, but think about the things you're trying to match. What were those things? S: Points and lines. I: Good, so you have to think about where the corner is on the scissors and where the corner is on the door. Now, I know you know where the corner is on the scissors, where's the corner on the door? [S points correctly.] Good! Now, where are the edges on the door? [S points to the edge of the door and the visible edge in the wall next to the door rather than the line within the door frame.] Is that where the door opens around? S: No. I: Where does the door open? [S runs her finger across the doorway opening.] So where are the edges that open? [S shows the imaginary edge.] Good, now do you think you can match the scissors and the door? [S correctly matches the scissors on the top of the doorway.]

The student initially focused on one particular feature (the opening) and did not attempt to match the other two (the vertex and the lines). In this case, it was the identification of the vertex and the imaginary line which led to a successful match. However, given the initial correct matching of opening size, the added focus on the lines and vertex suggest that all three angular components have been recognised.

Student 5. This student had successfully completed Stages 1-3 and was asked to match the scissors to the wheel. The interviewer opened the scissors to about 45°, set the red line on the wheel against the green line on the base (see Figure 1(d)), and asked the student to turn the wheel through the same amount. The student turned the wheel through almost a whole revolution.

> I: That was where it started, where the red line and green line match up. [I turns wheel back to starting point.] I want you to turn the wheel through the same amount as I have opened the scissors. S: No, I can't see how. I: That's where the wheel starts and I can turn the wheel around to make different size angles. [I demonstrates by turning it gradually.] S: Oh yeah. [S points to red mark on the wheel and turns the wheel to about 45°.] There. [S picks up scissors and places them on top of the wheel correctly matching the vertex and the arms in both situations.]

The gradual turning of the wheel appears to have been the trigger for the student to recognise the angle. When the student saw different angles in the wheel, she was able to identify all three critical angular attributes: the turning, the two lines, and the pivot point.

Student 6. This student initially made the irrelevant match between the scissors and the wheel shown in Figure 1(d): She placed the scissors at one corner of the base and turned the wheel so that the end points of the scissors pointed towards the two marker lines.

> I: Now, I am not quite sure what you are trying to match there. What are the things you have to try to match between the scissors and the wheel? S: The two lines. [S points to the markers on the wheel and base.] I: Now, is that the corner point on the wheel? S: No. I: Where is the corner point on the wheel? The point that it is turning through when it makes different angles? [S turns the wheel back to its starting point to match the two lines.] That's where it starts from and if you turn it around [S turns wheel] like that to a different angle, where's the corner point on the wheel? [S places scissors at the same place as before.] That's where it starts from [I takes wheel back to starting point], you can turn it around like that [I turns wheel placing his hand on it as he turns it.] S: Here? [S runs her finger on top of the wheel to correctly show the angle.] I: Can you use the straw to show me where the angle is that the wheel has turned? [S correctly shows it.] Yes, that's right. That was the starting point, the starting line.[I runs finger along that side of the straw]. That's the centre point. [I points to it]. That's the finishing line. [I runs finger along the other side of the straw and then turns the wheel back to the staring point.] So the question I asked you was, can you turn the wheel to the same angle that I've opened the scissors? [S correctly places the scissors on the wheel and turns it until the marking line is level with the blade of the scissors.]

The student seemed initially to view the turning of the wheel and the opening of the scissors as being determined by the end-points, even though some global notion of opening and turning was evident. The action of the interviewer in turning the wheel with his hand on it, along with the use of the straw, appears to have led the student to selectively attend to the pivot point of the wheel and to help her integrate the three angular attributes.

Discussion

The mixture of successful and unsuccessful outcomes indicates that the model proposed by Mitchelmore and White (2000a) provides an initial framework for teaching angles by abstraction, but it does require further refinement.

Our findings confirm the significance of the number of visible lines in physical angle situations, but caste doubt on the value of selectively attending to the individual attributes of the angle (point, lines, and opening) in a disjoint fashion. An over-emphasis on identifying the two lines in a physical angle situation may lead to superficial matching. This clearly happened in the cases of Student 2 (where his drawings indicated he was only matching the vertex and the lines and not the opening) and Student 3 (who seemed to be looking around for lines and a vertex to match, without any reference to opening). It seems necessary that all attributes be attended to simultaneously if angular matching is to occur. For example, Student 4 initially focused only on the opening; but when she was also directed to the vertex and the imaginary line, she appeared to make a match based on all three attributes.

The data strongly suggest that more attention needs to be given to within-context and even within-situation similarity recognition before attempting to match situations across contexts. For example:

- Student 2 could not match the angles in the fan and the scissors because he could not identify the vertex in the fan. He needs more experience with fans, in particular in analysing precisely what happens when they open varying amounts.
- When Student 5 saw the wheel turn through different angles, she understood more clearly how the wheel turned and was able to identify the vertex and lines of the angle of turn. Student 6 was helped in a similar way when the interviewer turned the wheel with his hand on it.

A key strategy in within-situation analysis is the use of drawing. For example, even though the final drawing of Student 2 may still only have been a drawing of the two lines and the vertex, the action of superimposing the scissors on the drawing did demonstrate to him the irrelevance of the length of the lines and drew his attention to the opening.

Mitchelmore and White (2000a) advocated the use of a bent straw to assist with similarity recognition. The data here are ambivalent about the value of this model. On the one hand, the straw helped Student 6 make an angular match. On the other hand, after correctly showing the angles in the scissors and the door with the straw, Student 1 could still not match the angle in the scissors and the door. White and Mitchelmore (2001) describe this as a *non-transitivity* effect; it was also observed in several students who could correctly match two angle situations with an abstract angle diagram but could not match the two situations. The straw is so decontextualised that it is virtually as abstract as an angle diagram. It may therefore only be of help where a student has practically made an abstraction already, by highlighting the critical features.

Implications for Teaching

Taken overall, the teaching sequence suggested by Mitchelmore and White (2000a)—and, by implication, the general method of Teaching for Abstraction proposed by Mitchelmore and White (2000b)— is supported. However, certain variations are suggested. In particular, the focus on identifying the two lines of an angle needs to be modified. It appears difficult for a student to identify angles without already having a "sense of angle" and this sense needs to come initially from a more global recognition of angle similarity before the critical attributes are identified analytically. In particular:

- Before matching different two-line situations, it might be effective to investigate different examples of the same situation. For example, students could search for matching corners in an assortment of tiles of different shapes, sizes, and colours.
- Similarly, a variety of opening and turning situations should be investigated before matching across situations. In particular, the two common methods of measuring the size of an opening—by an angle and by the distance between the end-points—need to be identified and distinguished. A match such as that shown in Figure 1(c) could then be accepted as a valid way of matching the openings of the

two objects; but an alternative match, based on angle, should also be readily apparent.

- Students should become more familiar with the characteristics of the separate contexts before they are directed to similarities between contexts. For example, before attempting to link scissors, doors, and wheels, students should look for similarities between (a) scissors, fans, and other hinged objects; (b) doors, windows, and other opening objects; and (c) wheels, knobs, and other turning objects. Recognition of these within-context links should be facilitated by the extra emphasis on within-situation similarities.

Further teaching experiments to test the revised sequence and approach in a more realistic classroom context are planned within the New South Wales *Count Me Into Space* project.

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