Visible and Invisible Zeros: Sources of Confusion in Decimal Notation

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Many researchers have used the task of comparing the size of pairs of decimals to reveal students' thinking about decimal notation. This paper reports data from 315 Japanese and Australian students on a modification of this task, comparing decimals involving zero. Many students think decimals are less than zero. Some students who otherwise appeared to be experts exhibited difficulties executing comparison strategies that involved adding zeros to the end of decimal strings. Students believing that the decimal number system is discrete were identified.

Many students have difficulties understanding the meaning of decimal numeration, a fact that is frequently reported in the literature and anecdotally. Recently a librarian has told us of the need to provide substantial training for new library staff in shelving books according to the Dewey decimal system, which requires an understanding of the order (although not in any meaningful sense, the size) of decimal numbers. On another occasion, a lecturer of statistics reported to us the difficulty that her students have when statistical software calculates a p-value for a statistical test. A student may know that p = 0.0493 but often does not know whether this is less than or greater than the cut-off value of p = 0.05.

One task that has proved to be very revealing is to compare the size of sets of decimals. Figure 1 shows one example of this task, given to a random sample of 13-14 year old students in TIMSS, the Third International Mathematics and Science Study. Data held at the Australian Council of Educational Research shows that Australia was a little above the international average (44%) on this item, with 47% of 13-14 year old students selecting (b) correctly. In Singapore, 84% of students were correct. Answer (a), which would be correct if the decimal points were interpreted as fraction lines, was selected by 25% of Australian students. Answer (c), in which the length of the decimal numbers increases (8 – 19 – 345), was chosen by 18% of students.

Which list shows smallest to largest?				
(a)	0.345	0.19	0.8	1/5
(b)	0.19	1/5	0.345	0.8
(c)	0.8	0.19	1/5	0.345
(d)	1/5	0.8	0.345	0.19

Figure 1. An item from TIMSS

For research purposes, carefully controlled sets of comparisons of the sizes of decimal numbers give more insight into how students are thinking about decimal numeration than an item as in Figure 1. Sackur-Grisvard and Leonard (1985) used comparisons of triples of decimals to identify three common misconceptions but in subsequent years, it has been established that comparisons of pairs is simpler for students, reducing information processing demands, yet can reveal as much (Resnick, Nesher, Leonard, Magone, Omanson & Peled, 1989).

In a series of papers (Stacey & Steinle, 1998, 1999; Steinle & Stacey, 1998) we have reported on the patterns of thinking about decimal numeration that we have discovered amongst a large sample of Australian students by using a Decimal Comparison Test (DCT). By examining the scores of students on groups of items, we can diagnose the way in which about 70% of students interpret decimal numeration. Our interviews with students and other observations of their behaviour (e.g. when playing computer games) confirm that the diagnosis is usually correct, although it is possible that students sometimes apply different ideas in different situations. The DCT contains items of 7 *item types*. Items of type 1, for example, require students to circle the larger of a pair of decimal numbers where the shorter decimal is the larger number e.g. 4.8 compared with 4.63. In items of type 4 one number is a truncation of the other e.g. 8.245 compared with 8.24563. Decimal comparisons belong to the same item type if individual students answer them consistently, whether correctly or incorrectly. Defining a type requires combining mathematically significant and psychological significant features of decimal comparisons. There is therefore a duality between item types and student conceptions. On the actual test, the order of the smaller and larger decimals is randomised. However, in this paper, we always give the larger of the pair first.

The pattern of responses to the groups of items of each type enables us to diagnose the way that a student is thinking about decimals. About a dozen patterns of thinking can be diagnosed with the 7 item types. The students who answered (a) to the item in Figure 1 above, would be very likely to score highly on type 1 and low on type 4. They interpret a decimal number as analogous in some way to a fraction. The exact analogy varies. Those who answered (c) would be likely to score low on type 1 and high on type 4. They may interpret the number after the decimal point as a number of parts of unspecified size. Again there are several variations not discussed here. We call students who score high on all our item types, apparent experts. They are not necessarily true experts because the DCT only reveals some aspects of students' thinking about decimals. There is much more to know, including, for example, the ability to place a number on a number line. Another reason for designating these students as apparent experts is that they may be following a rule for comparing decimals without understanding it. There are two expert rules for students to follow, which work on all comparison pairs. First, digits can be compared from left (tenths) to right until one digit is found to be larger than the corresponding one (so 0.345 > 0.32 because 3 = 3 and 4 > 2). Second, zeros can be added to the shorter decimal to equalise length and then the digit strings can be compared as whole numbers (so 0.3451 > 0.32 because 3451 > 3200).

The item types in the DCT mainly avoided the use of zero in the decimal part of the numbers. This decision was made initially to keep the test to a reasonable length, and it was judged better to cover the item types thoroughly. Because of the special role of zero, there are several different item types needed to fully explore it. It also requires the possibility of the pairs of numbers being equal (for items such as 0.8/0.80 as well as items such as 0.21/0.021, which students may believe are equal). This paper therefore reports a study to complete our investigation of how students interpret decimal numeration by extending it to cover their reactions to zero in several different roles in decimal numbers and to allow the pairs to be equal. The results are especially revealing about the understandings of the apparent expert group of students.

Method and Subjects

A new test, DCT0, the Decimal Comparison Zero Test was created, with the instruction: *put a ring around the bigger decimal number or write = between them*. Thirty comparison items appeared on one page and the first two illustrated what to do with an equal and unequal pair. From the 28 remaining test items, this paper analyses the results of 22 items arranged in Table 2 in what we had expected (with exceptions discussed below) to be the homogeneous item types. Item type 10 could not be asked before. Some item types are modifications (e.g. allowing or not allowing the same digits in types 9 and 15) and item type 4 is the same as before. The new possibility of designating the pair as equal enabled new investigations, as explained below.

The first sample taking DCT0 consisted of 104 teacher education students from an Australian university. This sample was given a sheet of paper with the DCT on one side and the DCT0 on the other. It is not known which side was completed first. The papers were marked and analysed by the first author who noted correctness and also any alterations to answers. One student did not complete the DCT test and another did not complete the DCT0 test, so that the error rates are based on 103 students. The second sample consisted of 212 school students in Japan: 36 students from Elementary School Grade 6, 33 from Middle School Grade 1, 102 from High School Grade 1 and 41 from High School Grade 3. This sample completed the DCT0 only. These papers were corrected in Japan (by Dr Keiko Hino), and the frequency with which students altered answers was not recorded.

Results

Table 1 gives the percentage of students in each sample with few (0 or 1) and many (more than 6) errors out of the 28 items. As expected, the Japanese sample improves with age. The Australian sample is rather like the Japanese MS Grade 1 group. Japanese students performed better on most of the items than Australian students.

Table 1
Percentages of students with few and many errors on DCT0 by sample.

	Aust. sample (n=103)	Japanese sample (n=212)	Japanese sample by grade			
			ES Gr6 (n=36)	MS Gr1 (n=33)	HS Gr1 (n=102)	HS Gr3 (n=41)
Few errors	62	71	36	58	81	88
Many errors	20	10	22	18	6	2

Table 2 shows the error rate for each item for both samples and it shows how the errors were made. The columns in Table 2 labelled “<” record the number of responses that (effectively) claimed that the first number is less than the second number (similarly for “>” and “=”). The similarity of error rates helps confirm that the items placed together are indeed of the same type, with exceptions discussed below.

Table 2
Overall Error Rates by Item and Nature of Error Within Each Sample

Details of items		Error rate (n= 315)	Japanese students (n = 212)		Aust. education students (n=103)	
			<	=	<	=
		Visible initial zeros: Type 9				
Q4	0.8 / 0.08	1.3%[a]	2	0	1	0
Q5	0.8 / 0.0008	1.3%[a]	2	0	1	0
Q10	0.45 / 0.045	1.6%	4	0	1	0
Q16	1.347 / 1.0347	3.2%	6	2	2	0
		Visible interspersed zeros: Type 15				
Q12	0.45 / 0.405	6.7%	16[b]	0	5[b]	0
Q20	1.347 / 1.30407	9.2%[a]	16[b]	1	8[b]	0
		Visible final zeros: Type 10				
Q3	0.8 / 0.80	6.7%	11	5	2	3
Q6	0.8 / 0.80000	8.6%[a]	8	12[b]	3	2
Q11	0.45 / 0.450	6.7%[a]	11	4	1	4
Q13	0.45 / 0.45000	6.3%	12	6	1	1
		Comparison with zero: Type 8				
Q21	1.3470 / 0	2.2%	6[b]	0	1	0
Q22	0.6 / 0	13.0%	23[b]	0	18	0
Q23	0.6 / 0.0	10.5%	18	0	15	0
Q24	0.6 / 0.000	9.5%	15	0	15	0
		Invisible zeros: Type 4				
Q7	8.41242 / 8.41	12.1%	13	3	18	4
Q14	0.73222 / 0.73	12.4%	14	3	17	5
Q27	2.884321 / 2.88	12.4%[a]	16	1	16	5
Q15	1.3470 / 1.3	11.4%[a]	12	3	15	5
Q8	6.12899 / 6.12	11.1%[a]	12	2	17	3
Q18	3.77777 / 3.7	11.7%[a]	11	3	13	8[b]
Q28	4.666 / 4.66	14.6%	20[b]	4	14	8[b]
		Base line error rate				
Q17	9.56 / 4.77	1.3%	2	0	2	0

[a] Items where there were some omissions (averaging 0.2%)

[b]Unexpected entries

The main differences between the Japanese and Australian samples are in item type 10, where the Australian students did better and in item types 4 and 8, where the Japanese students did better. These differences are discussed below. The last item in Table 2 (9.56/4.77) is an indicator of the “careless error” rate (1.3%), since there is no known reason that would lead students from Grade 6 and above to make a ‘systematic’ error on an item like this. Beyond the ‘careless errors’, a small group of students are generally responsible for a lot of the errors on any one item type and this is what gives us information about the way that students think about decimals. For example, 11 Japanese students had all four of the type 10 questions incorrect. Thus, the error rates are only the first data that we use to identify students’ thinking.

How Students Interpret Visible Zeros in a Number

Patterns of answers to the DCT0 enables the investigation of how students interpret the digit zero in critical positions in a decimal (in the tenths, on the far right and interspersed throughout other digits) and how students order decimals in comparison with zero itself.

The low error rates for item type 9 shows that almost all students know that “initial” zeros (i.e. in the tenths column etc) make a decimal part smaller. Our other investigations (Steinle et al, 1998) have lead us to believe that many students know this as an isolated fact, rather than as part of an integrated conceptual system for understanding decimal numeration. The difference in error rate between Q16 and the other three questions is unexplained.

We had expected type 10 items to be better done than type 9 items, since the reasoning that 0.80 = 8 tenths + 0 hundredths = 8 tenths = 0.8 seems very easy. Moreover, “adding a zeros to equalise length” is a taught expert strategy for comparing decimals. However type 10 was more difficult than type 9, especially for Japanese students. An examination of Japanese textbooks and a personal communication from Dr Keiko Hino indicates that this “adding zeros” expert strategy is not taught commonly in Japan. This lack of emphasis may explain the relative difficulty of these items for the Japanese students. (Note that this does not show “adding zeros” is the preferable strategy to teach because the Japanese students are better overall.) The four type 10 items behave uniformly, except that the longer decimals (especially 0.80000) are somewhat more likely to stimulate the judgement that it is a larger number.

The two items of type 15 with interspersed zeros (e.g., 0.45 / 0.405) were also more difficult than expected for both samples. The students do not think that these numbers are equal, so they know that the zeros are significant. Comparing the error rates for item type 1 on the DCT for the Australian sample showed that the error rate could not be explained for by a general tendency to choose longer decimals as larger. This may be worthy of further investigation.

Comparison of a number with zero: Type 8. A meteorologist presenting a weather report on ABC radio (April 3, 2001) reported that the minimum overnight temperature was “zero point four” degrees. The presenter commented “this is our first sub-zero temperature this year” but immediately corrected herself to “near zero”. When asked (by the first author) why she may have made this slip she suggested that it was “because the zero was there” and that possibly “point four would have been OK”. Items of type 8 tap into this phenomenon by comparing a decimal number with zero. Our previous work (Stacey, Helme, Steinle, Baturo, Irwin & Bana, in press) found that 13% of a student teacher sample thought decimal numbers are less than zero and that this is most pronounced when the comparison is with 0, rather than 0.0 or 0.000 etc.

The present results on DCT0 confirmed this, coincidentally with an overall error rate of 13%. Thirty students were incorrect on all three items Q22, Q23 and Q24. Further evidence of the difficulty of these items comes from the fact that 44 of the answers to these three items by the Australian students were altered, more than double the rate of alterations on any other type. Q22 (0.6/0) was the most altered item on the test.

The much reduced error rate for the comparison of 1.3470 with 0 was predicted from the explanation that we put forward (Stacey, Helme & Steinle, 2001) based on the underlying metaphorical thinking. Interviews indicated that some students perceive that 0 is in the units column whereas 0.6 is in the tenths column. This makes 0 a whole number, whereas 0.6 is a decimal and less than a whole number. With this thinking pattern, students see 1.3470 as starting in the units column, hence being bigger than zero. These students place numbers on a mental "number line" that is built on the place value columns (thousands, hundreds, tens, units, tenths, hundredths, etc). Future work might also use comparisons between 0.0 and 0 or 0.0 and 0.000.

How Students Deal With Invisible Zeros in a Number

In the above discussion, we have reported how the use of the equals option in the comparison task enabled a more thorough examination of students' interpretations of zero in a decimal number. Items of type 4 are used to examine a different phenomenon. A high error rate is expected on type 4 items because students with one of the short-is-larger misconceptions (e.g. answering (a) in Figure 1) will make errors. However, beyond this, we observed (Stacey & Steinle, 1998) that a significant number of students (about 5%) who were otherwise correct on all the item types on the DCT scored low on type 4. One of these students commented to us that she could not tell which of the numbers was larger, because they were both the same when read as amounts of money. For example, she tried to compare $17.35 with $17.353 and believed that these are the same. In fact, she imagined real numbers, like coins and notes, to be discrete, going only to hundredths. Beyond the second decimal place the numbers were meaningless, just as fractions of a cent are meaningless as individual amounts to pay. This belief may also have been reinforced by regular instructions throughout secondary school to round (or truncate) answers to 2 decimal places. She believed that rounding or truncation was required not because of consideration of the degree of accuracy appropriate in a question, but because the extra digits were completely meaningless – some sort of "error".

DCT0 enabled us to investigate whether such "money thinking" explained the type 4 errors. These students would say type 4 items are equal. In fact, since they have been taught at school about rounding, they may truncate or round, and Table 3 shows the answers they would then give on the 7 type 4 items of DCT0. Since we have also interviewed students who looked at only the first decimal place (not two places) to make a decision, the answers of students rounding or truncating in this way are also given.

Table 2 shows that type 4 was the only type to attract erroneous = answers and about 1% of the Japanese sample and about 5% of the teacher education sample do this on any one item. When we examined the answers of all the (Australian) students who scored highly on all the item types other than type 4 on both DCT and DCT0, we found 5 students (2 + 0 + 1 + 2) who completely followed the patterns for rounding and truncating shown in Table 3. This confirms that there is a group of students who, although they might seem to have excellent understanding of decimal numbers, actually have very little understanding at all. They operate successfully in a limited number of decimal places and believe that the number system is discrete.

Table 3
Answers Resulting From Possible Strategies of Apparent Experts on Type 4 Items

Item	Truncating		Rounding		Left to Right	
	1dp	2dp	1dp	2dp	Cf space	Cf zero
8.41242 / 8.41	=	=	=	=	G	>
0.73222 / 0.73	=	=	=	=	G	>
2.884321 / 2.88	=	=	=	=	G	>
1.3470 / 1.3	=	G,>	=	G,>	G	>
6.12899 / 6.12	=	=	=	>	G	>
3.77777 / 3.7	=	G,>	>	G,>	G	>
4.666 / 4.66	=	=	=	>	G	>
No. of students	2	0	1	2	5	many

Note. The correct answer to each item as presented in this table is “>”. Items marked “G” require dealing with invisible zero(s). If students cannot do this, they can only guess.

It is still left to explain why about 15% of the Australian sample and 8% of the Japanese sample chose < (rather than =) for each of the type 4 items. One reason would be that they consistently select shorter decimals as larger (as noted above), but the results of the DCT showed that only 2% of the Australian sample did this consistently, leaving most of the behaviour unexplained. One clue came from the very high rate of = on Q18 and Q28 (which needs further investigation). Some students may believe that 3.7 = 3.7777 = 3.7repeated. We did find one student (but only one) who used this logic nearly consistently on both DCT and DCT0, as if she had equated 8.41 with 8.411111 in order to compare it with 8.41242, 2.88 with 2.888888 to compare with 2.884321 etc. In sum, these explanations leave unexplained the behaviour of about 6% of the Australian sample, who answer items of other types mostly correctly.

Our current explanation is as follows. Following the expert strategy of equalising numbers by adding zeros leads to correct answers. Many students probably followed this strategy, especially the Australian students. One student, for example, equalised length by *writing* all the equalising zeros on all 7 items in type 4. The left to right digit comparison expert strategy, however, meets an unusual complication in type 4 items. To compare, for example, 8.41242 and 8.41, one checks 8=8, 4=4 and 1=1 and, since the decision is not yet made, 2 has to be compared with a space. The student using this strategy has to know to compare 2 with 0 (i.e. to consider as necessary 8.41, 8.410, 8.4100 etc). The invisible zero has to be made visible for this strategy to work. As Table 3 shows, if the student does know this, they are correct on all items, but if they cannot compare with the space, they have no option but to guess each item. Table 3 shows that we found 5 otherwise expert students whose answers over the whole test could fit this pattern. Note that item type 10 also has this property, in the absence of special knowledge of the terminal invisible zeros.

Conclusion

As one student wrote on a test paper, “zero is a very tricky number”. Extending the decimal comparison test to allow pairs to be judged as equal and including a variety of item types with visible zeros has shown several misunderstandings held by students who otherwise appear expert. These include basic place value equivalences (0.8 = 0.80 etc) and comparison of numbers with zero. Both the expert strategies for comparing decimal numbers require an understanding of how to work with the 'invisible' zeros waiting for action to the right of any decimal string. Evidence was found that a small number of students view the decimal number system as discrete, which probably arises from teaching which does not go beyond an analogy with money and too frequent rounding to 2 decimal places. These students may seem expert, but they actually have very little understanding.

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