A Framework for Algebraic Insight

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Using advanced calculators to do mathematics changes the focus of what students need to learn. For arithmetic number sense is important, for algebra, symbol sense. Both concepts apply to the whole problem solving cycle. This paper is concerned with algebraic insight, the subset of symbol sense which is needed to solve a problem already formulated mathematically. Students need algebraic insight to enter expressions correctly, monitor the solution process and interpret the output as conventional mathematics. A framework for algebraic insight is described and illustrated with examples.

The increasing availability of Computer Algebra Systems has given a sense of urgency to the question ‘If the calculator can do it, what do we need to teach?’ This paper presents a framework for looking at the mathematical thinking students need in order to ‘do algebra’ with CAS.

Number Sense, Symbol Sense

In 1982 Wilf alerted the *American Mathematical Monthly* to ‘The disk with the college education’. He had just spent an evening working with an early CAS and in his short article he raised a number of important questions which we have yet to answer. In particular he asked how the content of courses should be affected and whether we will “take the advice that we have been dispensing to teachers of the primary grades: that they should teach more of concepts and less of mechanics?” (p. 7) Since then it has commonly been stated that the computational capabilities of technology should change the emphasis of teaching in mathematics classrooms. While students struggle to manipulate numerical and algebraic symbols CAS can perform these tasks accurately and quickly. However as Fey (1990) pointed out

> Even if machines take over the bulk of computation, it remains important for users of those machines to plan correct operations and to interpret results intelligently. Planning calculations requires sound understanding of the meaning of operations – of the characteristics of actions that corresponds to various arithmetic operations. Interpretation of results requires judgement about the likelihood that the machine output is correct or that an error may have been made in data entry, choice of operations, or machine performance. (p. 79)

For arithmetic, this understanding and ability to both plan and interpret has been termed *number sense*. This concept of ‘number sense’, needed to work successfully in partnership with scientific calculators, has been discussed and written about since the late 1980’s and is well described by McIntosh, Reys and Reys (1992) For algebra, a logical extension of number sense is *symbol sense*. Some keys themes for teaching symbol sense were suggested by Fey (1990) and Arcarvi (1994). Arcarvi, reflecting on his teaching experience, suggested a list of attributes that are indicative of symbol sense. His list covers the thinking involved at all stages of mathematical problem solving including formulating the problem and interpreting the solution. CAS do not assist at all stages of problem solving, they only perform algorithmic processes. The contribution of CAS is in the mathematical solution of a formulated problem. The subset of symbol sense that enables this partnership to be effective is what we call *algebraic insight.* It is the algebraic knowledge and understanding which allows a student to correctly enter expressions into a

CAS, efficiently scan the working and results for possible errors, and interpret the output as conventional mathematics. Figure 1 shows this relationship, with symbol sense applying generally and algebraic insight applying only to solving the mathematical problem.

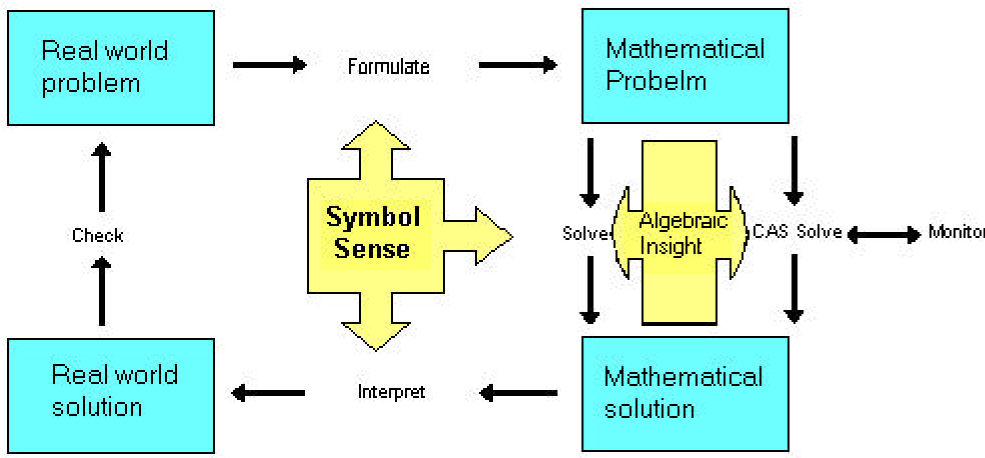


Figure 1. The place of symbol sense and algebraic insight in mathematical problem solving.

Algebraic Insight: A Framework

Algebraic insight is summarised in the framework presented in Table 1. First it should be emphasised that this is a working framework, a basis from which to plan, assess or reflect on students' *algebraic insight*. The elements of each aspect of *algebraic insight* can be used as the springboard for developing test items or examples but a particular test item may involve more than one element of *algebraic insight*. The framework was developed in response to the literature and the first author's experience of teaching with CAS (Pierce & Kendal, 1999; Pierce & Stacey, in press). It is an attempt to make an analysis of what it is that 'expert' mathematicians do when they monitor mathematical working. Algebraic insight does *not* require carrying out manipulations and calculations of more than one step since this would defeat the purpose of using CAS.

Algebraic insight may be divided into two aspects. This is a logical division, considering first the insight needed working within a symbolic representation, and then the insight needed to work with and gained from linking the symbolic to graphic and numeric representations. The framework is restricted to these three representations since this is what CAS offer. The third possible link of information between the numeric and graphic representations is not discussed because our focus in on symbolic algebra.

Each aspect of algebraic insight has been divided into key elements and the common instances indicate when these abilities may be demonstrated. While the aspects and their elements have general application the common instances, through which algebraic insight is demonstrated, are specific to both stage and topic and the list of common instances is not exhaustive.

The framework is explained and illustrated below using common instances from a study undertaken in 1999, with undergraduate students in an 'introduction to calculus' course. Two instruments were developed to assess algebraic insight: a quiz and an interview. Details of the quiz were similar to those described by Ball, Stacey & Pierce (2001). At individual interviews, which were audiotaped, students were given a series of five questions to work ‘aloud’. Examples (see Figures 2, 3 and 4) and results from the quiz and interview (Figure 5) will be also discussed in relation to each of the elements of the framework.

Table1
Algebraic Insight Framework

Aspects	Elements	Common Instances
1. Algebraic expectation	1.1 Recognition of conventions and basic properties	1.1.1 Know meaning of symbols 1.1.2 Know order of operations 1.1.3 Know properties of operations
	1.2 Identification of structure	1.2.1 Identify objects 1.2.2 Identify strategic groups of components 1.2.3 Recognise simple factors
	1.3 Identification of key features	1.3.1 Identify form 1.3.2 Identify dominant term 1.3.3 Link form to solution type
2. Ability to link representations	2.1 Linking of symbolic and graphic representations	2.1.1 Link form to shape 2.1.2 Link key features to likely position 2.1.3 Link key features to intercepts and asymptotes
	2.2 Linking of symbolic and numeric representations	2.2.1 Link number patterns or type to form 2.2.2 Link key features to suitable increment for table 2.2.3 Link key features to critical intervals of table

Aspects of Algebraic Insight: 1. Algebraic Expectation

The term *algebraic expectation* is used here to name the thinking process which takes place when an experienced mathematician ponders the result they expect to obtain as the outcome of some algebraic process. Skill in algebraic expectation will allow a student to scan CAS output for likely errors, recognise equivalent expressions and make sense of long complicated results. It parallels arithmetic estimation which may involve estimating that the answer to 6283×4816 will be in millions; algebraic expectation may involve expecting that $(2 - x + x^2)(x^4 - x^3 + 27x - 63)$ will be a polynomial of degree six. Algebraic expectation does not involve producing an approximate solution but rather noticing conventions, structure and key features of an expression that determine features which may be *expected* in the solution. For this reason we have chosen the word *expectation* rather than *estimation*. These skills form the elements of algebraic expectation illustrated in turn below.

1.1 Recognition of Conventions and Basic Properties

Students must recognise the conventional meaning of symbols used in algebra (1.1.1). This involves both operators and 'letters'. While the operators +, -, and × should be familiar from arithmetic the convention in pen and paper algebra of implicit multiplication, where *xy* means *x times y*, is a source of confusion. There are further issues here that relate to the use of CAS. The required syntax is CAS specific but since CAS allow variable names to have more than one letter a distinction must be made between a variable *ab* and *a*b*.

Letters are used in a number of ways in algebra. For example a classic quadratic function is commonly expressed as $y = ax^2 + bx + c$. This requires a student to recognise that *a*, *b* and *c* are parameters while *x* and *y* are variables, two different meaning for letters in one statement. Students' understanding, or as MacGregor and Stacey (1997) report often misunderstanding, of the meaning of letters in mathematics is important for algebra with or without the use of CAS. One particularly interesting example from the interviews was shown in the work of Henry who followed the structure for item 4 (Fig 4) with numbers but when faced with part (ii) said confidently that the x values would be b, d and f since b is even (drawing on alphabetical order). Henry did not understand the meaning of letters as variables. The use of CAS raises some specific issues for understanding the meaning of symbols and their use in defining variables and parameters within the program. For example: variable names may be short words not just single letters and names, representing parameters, may be assigned specific values that will be used by the program instead of the letter until the assignment is changed or cancelled.

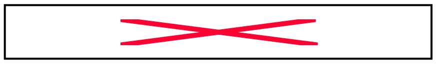
CAS programs can vary in the way they interpret syntax but most consider bracketed expressions first then carry out operations working from left to right. It is therefore necessary to use brackets to make the meaning of expressions clear and ensure that operations are carried out in the order expected. To ensure the conventional order of operations (1.1.2.) to the expression $a+p\div q$ it may have to be entered as $a+(p/q)$.

As well as the meaning of symbols and order of operations students need to recognise the basic properties of operations (1.1.3) for example that x ü y ? y ü x (question 4, figure 2) or that some operations are distributive others are not. This property is a common source of confusion for students that causes incorrect *algebraic expectation*s. For example students learn that $a(b+c) = ab+ac$ and then *incorrectly* expect that $(a+b)^2 = a^2 + b^2$.

Knowing the meaning of symbols, order of operations and properties of operations are all common instances when students demonstrate the first element of algebraic expectation. This ability is especially important for entering and interpreting CAS syntax. The second element of algebraic expectation is the identification of structure. One of the claims made for the use of CAS is that it allows students to tackle more realistic problems involving complicated expressions. Identifying structure helps students to check and interpret such expressions.

1.2 Identification of Structure

Consider the structure of the expression $\frac{a(x+1)^5 + b(x+1)^2}{(x+1)}$. The vinculum indicates the first level of structure in this expression that this is a division with a numerator and denominator. Viewed at another level (x +1) can be identified as an object which is common to each term of the expression. Recognising structure (1.2.1) can mean seeing at a glance that $2x+1$ is a common factor, for $(2x+1)^2 - 3x(2x+1)$ but looking at $(2x+1)^2 - 3x(2x+5)$ and noting that the bracketed objects differ.

A further common instance of identification of structure is demonstrated when students identify strategic groups of components. This helps the recognition of equivalent expressions, for example, $\sin^2 x + 2\sin x\cos x + \cos^2 x$ may be strategically grouped in two ways, as  and hence recognised as sin $(2x)+1$ or viewed as being of the form of a perfect square and so written as $(\sin x + \cos x)^2$.

Identifying objects (1.2.1), strategic groupings (1.2.2), and simple factors (1.2.3) are all ways in which students can demonstrate an identification of structure in an expression. A structural view of expressions will inform algebraic expectation as will the third element, the identification of key features. Key features, for example may alert the student to the 'family' to which an expression belongs and hence the properties to expect.

1.3 Identification of Key Features

For functions identification of key features may lead to expectation about number of solutions, solution type, number of maxima and minima, domain and range. Identifying form (1.3.1) for functions and hence applying associated knowledge, can mean noting that $2 + e^x$ is exponential and that $e^{2x} + e^x - 2$ is a quadratic in e^x. Identifying the dominant term (1.3.2), means for example, noting that 3 is the highest power of the polynomial, $x^3 + 5x + 1$. This indicates that there will be at least one and at most three distinct solutions for $f(x) = 0$. Key features provide information on which to base expectations or suspect errors. In question 10, figure 2, for example, identifying the key feature of the highest power of x in both expressions (x^5 for the student, x^4 for the text book) immediately signals that there is an error here; these expression are not equivalent.

4.	Student	Textbook	**10.**	Student	Textbook
	$x \div y$	$y \div x$		$x^5 - y^5$	$(x^2+y^2)(x-y)(x+y)$

Figure 2. Sample questions from quick quiz assessing recognition of equivalence.

Aspects of Algebraic Insight: 2. Ability to Link Representations

While recognising that the meaning of symbols, identifying structure and key features provide a great deal of information on which to base algebraic expectations more algebraic insight can be gained by linking this algebraic information to the corresponding graphical and numeric representations of functions. When using CAS, students can swap between algebraic, numeric and graphical windows at the press of buttons. This may be used to advantage in examining students' *algebraic insight*. Fey (1990) noted that understanding of algebra would be demonstrated by students' ability to move between algebraic, numeric and graphical representations of expressions. Students would demonstrate *algebraic insight* by being able to make estimates, have a rough idea what a graph of a function may or may not look like, have some idea of the sort of range of values that might be expected for a given domain. To show *algebraic insight* a student would, for example, not be expected to complete a least squares linear regression but to recognise that a set of data points might be modelled by a linear function. They would be looking for a function or class of functions to link between the dependent and independent variables. Similarly a student would immediately recognise a quadratic function rule as being represented by a parabolic graph and as Arcavi (1994) suggests the student may realise that each representation is equally valid and that changing representations may help them make progress in solving a problem.

2.1 Linking of Symbolic and Graphic Representations

Linking symbolic and graphic representations of functions involves linking, form to shape and key features to likely position, intercepts and asymptotes. Linking of shape to form (2.1.1) is shown when a student looks at a function like $p(t) = 5t^2 + 3t + 1054 + \sin(2\pi t + 5)$ and recognises that this is made up of a quadratic plus a sine function so its graph will look like a sine graph 'bent' up to follow the curve of the quadratic. In general, identifying form gives enough information about a graph to be able to draw the basic shape 'in the air' with a hand wave.

In the individual interviews, after a glance at the quadratic function in question 5, figure 4, each student responded with a U-shaped wave and identified –21 as a key feature (2.1.3) that indicated the *y*-intercept. Henry was typical of the group for whom this style of question was familiar from school mathematics. He answered quickly and correctly by identifying key features, in this case the coefficients 1, 4, -21 and working with these.

In question 21, Figure 3, for example, the student was expected to either identify form (2.1.1) or identify key features (2.1.2) by noting that the graph has 3 *x*-intercepts, or identifying that it has two turning points and so will be at least cubic. This makes alternative (d) the only possible choice and also demonstrates that algebraic insight could be shown here through several common instances.

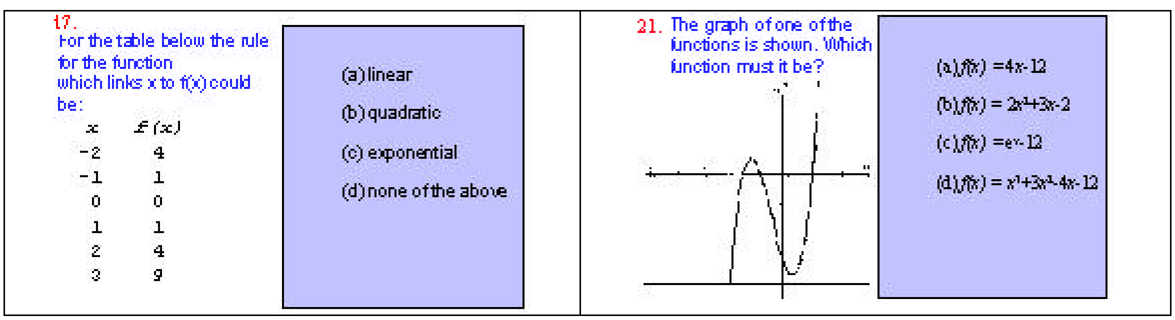


Figure 3. Sample questions from quick quiz assessing linking of representations

Algebraic insight will also provide information about the numeric representation of functions and provide a guide to the increment and interval that will reveal critical aspects of a function.

2.2 Linking of Symbolic and Numeric Representations

The ability to link symbolic and numeric representations will be shown when students link number patterns to form (2.2.1) and link key features to suitable increments (2.2.2) and critical values (2.2.3) of tables. Patterns in tables such as equal differences or equal ratios alert the mathematician to the likely form of the algebraic representation of these functions. In question 17, figure 3 the student might scan the relationship between x and $f(x)$ and notice the 2 maps to 4, 3 to 9, x to x^2. Alternatively the student might recognise that the $f(x)$ values are all squares.

Algebraic expectation will help the student choose a suitable increment for the independent variable. When students work by hand they commonly choose an integer increment and an interval around zero and so miss important features of the function. If, for example, $r(\theta) = \sin(2\pi\theta)$ using an integer increment for a table of values will result in 0 for each $r(\theta)$, not very helpful! The key feature to identify here is the 2π factor since any angle that is a multiple of 2π has a sine of zero. Recognition that the function will have repeating cycles of values suggests that a detailed representation of one cycle may give useful information about the function. CAS make using any increment or interval equally easy so with algebraic insight students can use this facility to gain more understanding of the salient aspects of a function.

4. If $T(x, x^2, 0,1)$ generates the table

x	f(x)
0	0
1	1
2	4
3	9

and $T(x, x^2, 0, 2)$ generates

x	$f(x)$
0	0
2	4
4	16
6	36

what will be generated by

$T(x, 3x^2, 0\ 1)$ $\quad$ $T(x, x^2, 0, b)$

$T(x, x + a, 2, 1)$ $\quad$ $T(y, y^2, 0, 2)$

5. $g(x)=x^2 +4x-21$

What parts of this rule immediately give you clues to the shape and position of a graph of this function? Explain.

Rewrite this rule so that the x intercepts of its graph can be easily seen.

Rewrite this rule so that the coordinates of the minimum point of its graph can be can be easily seen.

Figure 4. Sample questions from interview assessing multiple elements of algebraic insight.

Strengths and Weaknesses

This section presents a very brief overview of one of the ways in which we are using the algebraic insight framework to understand students' strengths and weaknesses. The overall class results for the quick quiz (pre and late course in the 1999 study) are shown in figure 5. The pre-course results reflect the students' school mathematics experience. There were clear weaknesses in their backgrounds especially in identification of structure (1.2) and key features (1.3) and working with tables (2.2). After the course (taught with CAS) there was noticeable improvement in all elements with most change in identification of key features (1.3) which was emphasised in the teaching. Targeted, guided exploration with CAS allowed students to look at key features (1.3) in many examples using the three representations. Similar exercises, using only CAS algebra, focused on structure (1.2) in algebraic fractions and composite functions but these produced more limited improvement.

Overall the interviews confirmed the findings of the quiz but they uncovered further examples of lack of algebraic insight. It was interesting to see, for example, that Henry could have such fundamental gaps in algebraic expectation (specifically 1.1.1 meaning of symbols) and yet be able to link representations. It was clear that he had been taught how to identify the key features that link quadratic functions and parabolas. Some elements and common instances of algebraic insight are clearly harder to learn than others. This framework provides a structure from which to identify these and so sets the challenge of developing more effective teaching strategies for these skills needed for working with CAS.

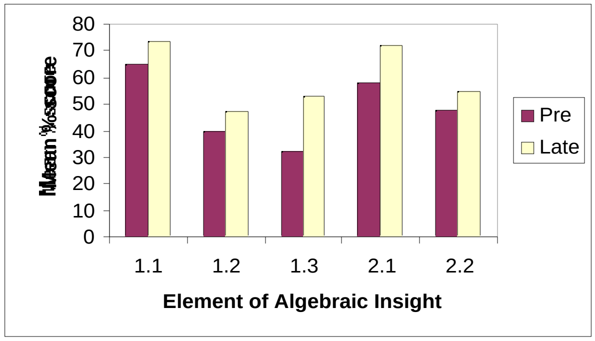


Figure 5. Mean percentage score (pre and post course) for each framework element.

Conclusion

The algebraic insight framework provides a helpful structure for examining the thinking required by students using CAS. The list of elements defines areas to be given attention in teaching or assessment. The framework enabled the test items to be grouped usefully to reveal strengths and weaknesses in students' algebraic insight.

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